

Global Structure of Five-dimensional BPS Fuzzballs

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Abstract

We describe and study families of BPS microstate geometries, namely, smooth, horizonless asymptotically-flat solutions to supergravity. We examine these solutions from the perspective of earlier attempts to find solitonic solutions in gravity and show how the microstate geometries circumvent the earlier “No-Go” theorems. In particular, we re-analyse the Smarr formula and show how it must be modified in the presence of non-trivial second homology. This, combined with the supergravity Chern-Simons terms, allows the existence of rich classes of BPS, globally hyperbolic, asymptotically flat, microstate geometries whose spatial topology is the connected sum of N copies of $S^2 \times S^2$ with a “point at infinity” removed. These solutions also exhibit “evanescent ergo-regions,” that is, the non-space-like Killing vector guaranteed by supersymmetry is time-like everywhere except on time-like hypersurfaces (ergo-surfaces) where the Killing vector becomes null. As a by-product of our work, we are able to resolve the puzzle of why some regular soliton solutions violate the BPS bound: their spacetimes do not admit a spin structure.

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1 Introduction: Six Impossible things before breakfast

Alice laughed. “There’s no use trying,” she said: “one can’t believe impossible things.”

“I daresay you haven’t had much practice,” said the Queen. “When I was your age, I always did it for half-an-hour a day. Why, sometimes I’ve believed as many as six impossible things before breakfast”

Through the Looking-Glass, Lewis Carroll

Much work over the last eight years has been devoted to constructing candidate ‘microstate geometries,’ sometimes referred to as ‘supergravity fuzzballs,’ with a view to resolving some long-standing puzzles in the quantum theory of black holes. The original idea of fuzzballs as a possible semi-classical geometric description of black-hole microstates originated from the work of Mathur and collaborators on two-charge systems. (For reviews of work on the two-charge systems, see [1, 2, 3].) In the F1-P duality frame, the states of the two-charge BPS system can be described by momentum modes on a fundamental string and so can easily be analysed and counted within the underlying conformal field theory. In supergravity, these solutions can be given a semi-classical description in which the string source with momentum waves is represented by an arbitrary shape profile for the string source. In addition to the F1-P electric charges, there is also a dipolar F1 charge and angular momentum along the string profile. If one transforms this to the duality frame in which it has D1-D5 charges then the dipole charge becomes that of a Kaluza-Klein monopole (KKM). The corresponding supergravity solution encodes that same semi-classical data as its F1-P counterpart but is, in fact, completely smooth in six dimensions [4, 5]. These are the original, prototypical microstate geometries: they are smooth, horizonless, asymptotically flat solutions with the same boundary conditions at infinity as a two-charge black hole. Mathur and collaborators went on to demonstrate that these fuzzball geometries could reproduce many of the properties of BPS black holes and of near-BPS black holes (see [6, 5, 1] for further discussion of this).

The problem with the two-charge system is that the corresponding black holes do not have a classical horizon: they only have a “stretched” or “effective horizon” at the Planck or string scale. This means that the smooth, classical configurations that are supposed to account for the black-hole entropy are at, or very near, the limit of the supergravity approximation. In spite of this, Sen has made some very interesting links between the microstate counting and the Bekenstein-Hawking entropy for the two-charge system [7]. This work was thus extremely suggestive but it really required the later analysis of Strominger and Vafa [8] to make a more compelling connection, at least at vanishing string coupling. The important difference was that Strominger and Vafa did the analysis for the three-charge black hole in five dimensions whose horizon area is much larger than the Planck scale. Similarly, the work of Mathur and collaborators on two-charge microstate geometries is remarkably detailed, very interesting and suggestive but it really requires the development of the three-charge microstate geometries in order to have

confidence in the validity of the supergravity approximation.

The first smooth, three-charge BPS microstate geometries in which all three charges are large were constructed in [9, 10, 11]. These solutions replaced the singular electric sources by smooth topological fluxes. They could also be given a rather attractive interpretation in terms of brane sources that have undergone a geometric transition. The systematic study of these solutions, their generalizations and the classes of non-BPS counterparts been a very active and fruitful area of research in the last few years.

Perhaps the simplest of these microstate geometries are asymptotically flat solutions of the equations of five-dimensional supergravity theories. However, there are richer classes of candidate microstate solutions that suffer from singularities when considered as five-dimensional spacetimes but yield smooth, non-singular, but no longer asymptotically flat, solutions of supergravity theories when lifted six space-time dimensions.

The five-dimensional microstate geometries represent a remarkable class of smooth, geodesically complete, asymptotically flat, stable spacetimes without horizons [9, 10, 11, 12]. (For reviews of work on the three-charge systems, see [13, 14, 15, 16].) Their existence is rather surprising for a number of reasons. First, one might think of them as the gravitational analogue of solitons in flat-space field theories, and no such solutions are known in four-dimensional supergravity. The best approximation to the soliton concept is the set of black-hole solutions that are non-singular outside an ‘extreme’ or ‘degenerate’ event horizon but which harbour singularities inside. Going back to the earliest days of General Relativity, there are a number of ‘No-Go’ results [17, 18, 19, 20, 21], which exclude completely non-singular soliton solutions that are regular in four spacetime dimensions.

For stationary metrics, one can establish no-go theorems¹ using the Smarr formula, which relates the Komar mass to densities that are to be integrated over interior boundaries. We will review these ideas in Section 5. These boundaries are usually either singularities or horizons and so, if one has a smooth, horizonless, asymptotically-flat solution one might naively expect that the mass must be zero and hence the solution is necessarily trivial. As we will discuss in some detail, five-dimensional supergravity is able to evade this conclusion because it has Chern-Simons interactions and the spatial sections may have non-trivial second homology. As a result, the one-form potentials may not be globally defined leading to extra bulk terms in the Smarr formula.

Thus microstate geometries require non-trivial homology in the space-time. BPS, or supersymmetric, geometries are necessarily stationary, with a non-space-like Killing vector, which however can be null on hypersurfaces. Supersymmetry also requires that the metric, $h_{\mu\nu}$, restricted to directions orthogonal to the Killing vector must be conformal to a hyper-Kähler metric, $\hat{h}_{\mu\nu}$.

¹This is the first *impossible thing*.

Hyper-Kähler metrics are necessarily Ricci flat. This presents a further challenge because a theorem of Schoen, Yau [22] and Witten [23] states that the only complete, non-singular, Ricci-flat, Riemannian manifold that is asymptotically Euclidean is, in fact, Euclidean space. In particular, there is no topology². Again, five-dimensional supergravity dodges this bullet and does so because the condition that the complete space-time must be a smooth, Lorentzian five-manifold is weaker than requiring that the spatial base, \mathcal{B} , equipped with either $h_{\mu\nu}$ or $\hat{h}_{\mu\nu}$ also be a complete, non-singular, asymptotically-Euclidean, Riemannian manifold. The base \mathcal{B} does in fact admit a complete, non-singular asymptotically-Euclidean, Riemannian metric but it is neither $h_{\mu\nu}$ nor $\hat{h}_{\mu\nu}$ and it is not Ricci flat.

Since regularity of the five-dimensional, Lorentzian manifold does not require that the hyper-Kähler metric, $\hat{h}_{\mu\nu}$ to be positive-definite, one may allow it to be apparently pathological: It can be ‘ambi-polar,’ that is, the signature can change from $+4$ to -4 with apparently singular intervening surfaces. The miracle is that the ‘warp factors’ of the time fibration and the angular momentum vector can convert this into a smooth, Lorentzian five-manifold. The hypersurfaces on which the signature of $\hat{h}_{\mu\nu}$ changes sign correspond to places where the Killing field becomes null. Away from these hypersurfaces the conformally related metric, $h_{\mu\nu}$, is positive definite but becomes singular when the Killing vector becomes null. In contrast to the usual situation in which this surface is a null hypersurface and thus a Killing horizon, in the present situation it is a timelike hypersurface. In fact, this surface is a novel, and hitherto unencountered, phenomenon of an ergo-surface with no bulk ergo-region: an evanescent ergo-region.

From the perspective of string theory, microstate geometries also seemed an impossibility³ within the validity of any supergravity approximation. Fifteen years ago, Horowitz and Polchinski [24, 25] observed that if one started from vanishing string coupling then stringy states, like all normal matter, will become more and more compressed as the string, and hence gravitational, coupling increases. On the other hand, the size of a black-hole event horizon increases with the gravitational coupling and thus one should expect that the perturbative states that were counted in [8] and seem to account for the entropy of a BPS black hole must necessarily be inside a horizon at any finite value of the string coupling. Again supergravity evades this conclusion because D-branes are solitonic objects whose tension is proportional to g_s^{-1} and so they become floppier and floppier as the string coupling increases. In particular, putting momentum and angular momentum modes on the right combination of D-branes can result in a supergravity configuration that grows at exactly the same rate as the would-be horizon size grows with g_s .

In five dimensions, a BPS black hole or black ring can only have a macroscopic horizon if it has ‘three charges’ and the horizon area then grows as $\sqrt{Q_1 Q_2 Q_3}$. While one can perform the

²This is the second *impossible thing*.

³This is the third *impossible thing*.

perturbative state counting for many D-brane systems, it was the three-charge problem that was of crucial importance precisely because it has a macroscopic horizon and thus the microscopic entropy of the perturbative system can be compared to the area of the horizon within a valid supergravity approximation. It is also precisely for such black holes that the Chern Simons terms become non-trivial and the microstate geometries scale with g_s in the same way as the horizon. However, activating the Chern-Simons form leads to another, possibly daunting, practical issue: This term has the form $F \wedge F \wedge A$, rendering the generalized Maxwell equations non-linear. Thus microstate geometries might have existed but analytic solutions could have proved impossible to find⁴. Even if some solutions could have been found, or an existence theorem proven, the non-linearities could have made the phase space structure impossibly difficult to analyze.

However, yet another miracle happens: The supergravity equations remain linear provided that they are solved in the correct order [26]. One has to solve a sequence of linear equations, essentially involving only the Laplacian on the four-dimensional spatial base, and the non-linearities of the Chern-Simons terms only generate quadratic source terms assembled from the solutions of earlier equations in the linear system. Thus, once one has found a suitable ambi-polar, hyper-Kähler base everything is linear and families of solutions can be assembled by superposition.

The only delicate part of the whole procedure is to make sure the ultimate solution is causally well-behaved. In particular, this involves making sure that there are no closed time-like curves. To do this one constructs complete four-dimensional hypersurfaces of constant time on which the induced metric is positive definite and everywhere non-singular. This provides the solutions with a global Cauchy surface and precludes the existence of closed time-like curves. Technically, this involves imposing algebraic relationships between the cohomological fluxes and the moduli of the hyper-Kähler metric. Intuitively, these algebraic relationships, or ‘bubble equations,’ express the fact that there is a balance between the gravitational force that tends to contract topological cycles and the fluxes that tend to expand the cycles. More generally, there are many examples of smooth geometries that can be made in this way, but a ‘randomly assembled’ solution is most likely to be pathological. As yet there are no general theorems about when smooth, causally well-behaved solutions can be constructed but some heuristic suggestions were made in [13] and a much more specific proposal, called the “split attractor conjecture,” was proposed and investigated in [27, 28].

The first three-charge microstate geometries to be constructed with all three charges large were still extremely specialized: they carried the same quantum numbers as a black-hole of zero horizon-area [29]. That is, they typically had maximal angular momentum and the microstates described by these geometries appeared to be only marginally bound. It became an open question

⁴This is the fourth *impossible thing*.

as to whether any of these solitonic solutions could be arranged to look, asymptotically, like a true BPS black hole with what appears, from a distance, to be a macroscopic throat. The solution to this problem came [30, 31, 32] from the class of microstate geometries called “scaling solutions.” From the perspective of the spatial base geometry, this appears to be a singular limit in which a cluster of homology cycles are blown down while the fluxes on them remain finite. However, from the perspective of the five-dimensional geometry this limit corresponds to the opening of an AdS throat and the smaller the cycles in the base geometry, the deeper the throat. In this scaling geometry, the homology cycles limit to a finite size and the cross-section of the throat becomes a sphere whose radius is determined by the fluxes on the cycles that are scaling. Thus the full geometry looks exactly like the AdS throat of a typical extremal black hole or black ring except that it ‘caps off’ smoothly at some depth, determined by a scaling parameter, and the cap consists of finite-sized homology cycles carrying fluxes. In this way, the microstate geometries have a scale that is set by the horizon size of the black hole.

If a five-dimensional microstate geometry has a $U(1)$ symmetry then it can be reduced to a four dimensions and it generically becomes a four-dimensional, multi-centered black-hole solution. The black-hole centers are located at the fixed points of the $U(1)$ action and the corresponding sources may be thought of as fluxed D6-branes [33]. These solutions play a very important role in the counting of black-hole entropy at weak coupling using quiver quantum mechanics [31]. Indeed, it was in this context that scaling solutions were first discovered and investigated [27, 34, 35] but it was not until later that the significance of scaling solutions for five-dimensional microstate geometries was discovered [30, 31, 32].

Classically, the depth of the AdS throat in a five-dimensional scaling solution is set by a free parameter but semi-classical quantization of the moduli space of solutions limits the depth of the throat in a manner that depends upon the charges. This has the interesting consequence that quantum effects are wiping out large scale naively-classical geometries⁵: the scales of much deeper throats are classically large, certainly much larger than Planck scale, and the supergravity approximation remains valid and yet quantum effects are removing such macroscopic structures [32, 36, 37].

Another of the apparently impossible things that emerged from these families of microstate geometries is the extent to which they might be “typical.” Obviously, from the perspective of quantum mechanics, any large classical configuration is a very coherent state but a more important issue is whether it is a state from a highly atypical sector of the quantum theory. The fact that these solutions have a very long AdS throat means that this issue can be addressed directly by using the AdS/CFT correspondence [38, 39, 14]. Indeed each smooth bubbled geometry with a long AdS throat must be dual to a state in the black-hole CFT that underpinned the original

⁵This is the fifth *impossible thing*.

state counting in [8]. One can compute the lowest energy fluctuations of a scaling geometry by looking at the longest wavelength fluctuations that can be accommodated within the throat and then use the red-shifts generated by the depth that is set by the semi-classical quantization of the geometry. One finds [30, 32, 36] that these lowest energy fluctuations have the same energy as the fundamental fluctuations of the dual CFT that was used to count the states of the black hole [8]. Thus the microstate geometries have deep AdS throats and are dual to states in the “typical sector” of the conformal field theory⁶ and do not seem to represent unusual outlier states in the overall ensemble.

In conclusion, despite many potential obstructions, there are indeed viable BPS microstate geometries that are, by definition, smooth, horizonless solutions lying within the validity of the supergravity approximation to string theory and that look exactly like BPS black holes and black rings until one gets arbitrarily close to the horizon. The purpose of the present paper is to give a brief introduction to these remarkable solutions and then provide a global analysis, focussing on their topology and how the construction evades the various geometrical threats to their existence.

The plan of the paper is as follows. In Section 2 we provide some historical context for our work by reviewing some of the earlier attempts to find solitons in field theory. In Section 3 we discuss issues of global structure and causality in five dimensions and we discuss the requirements on the metric that guarantee that the space-time is non-singular and globally hyperbolic and we give a simple example of an evanescent ergo-region. In Section 4, we introduce the $\mathcal{N} = 2$ supergravity theory, coupled to two vector multiplets, that will be the basis of our analysis and we describe the metric Ansatz and BPS equations that are appropriate to the study of geometries that preserve the same supersymmetries as a BPS black hole. Section 5 contains a careful analysis of the Smarr formula for $\mathcal{N} = 2$ supergravity in five dimensions and, in particular, we examine precisely how topological contributions can arise in the Smarr formula and thus how microstate geometries can evade the “no-go” theorems. In Section 6 we review families of microstate geometries and analyze two examples in detail. Another non-BPS example is discussed in Section 7. Using the methods developed earlier, we are able to resolve the puzzle why some regular soliton solutions violate the BPS bound. It turns out that these particular non-BPS space-times do not admit a spin structure. The topology of the BPS solutions is discussed in Section 8 and our conclusions may be found in Section 9.

2 A brief history of solitons versus particles

In our introduction we reviewed the history of microstate geometries within string theory. In this section we shall place this within the general context of efforts to replace point particles by

⁶This is the sixth *impossible thing*.

smooth, essentially geometrical configurations, now often called solitons. In this way we hope to set the results described in the introduction and the results of the present paper in the context of a great deal of activity going back to at least the beginning of the last century: the aim being to replace Boscovich’s eighteenth century vision of point particles acted upon by the fields they generate [40].

This programme became more urgent with the rise of classical electron theory [41] and the realisation that the self-energies of electrons, modelled as classical point particles, contributes to their inertia, but is divergent. Even if this problem could be circumvented, there remained the objection that in such theories of particles and fields, the equations of motion for the particles and for the fields need to be postulated separately.

Gustav Mie [42], in an attempt to solve both problems turned to non-linear theories of electrodynamics, in which the classical self-energy

$$\int \frac{1}{2} \mathbf{D} \cdot \mathbf{E} d^3 x$$

of charged particles could be finite because the electric displacement \mathbf{D} is a non-linear function of the electric field \mathbf{E} . Another consequence of the non-linearity was the possibility that the equations of motion of the finite-energy point singularities might follow from the non-linear field equations without the need to be postulated separately, thus arriving at what was then referred to as a “Unitary Theory.”

With the advent of quantum mechanics, these attempts were, for a while, abandoned but the development of Quantum Electrodynamics and the discovery that vacuum fluctuations led to both divergent self-energy and non-linear corrections to Maxwell’s equations motivated Born to return to them, and develop the most attractive version of the ideas now known as Born-Infeld theory [43]. Born’s programme continues to attract interest in its own right not least because of its relation to strings ending on p-branes [44, 45, 46, 47, 48, 49]. It appears that the question whether the equations motion of its singular but finite energy classical point particle solutions (Bions⁷) follow rigourously from the equations of motion is still open [52, 53, 54, 55, 56, 57, 58, 59, 60].

Einstein’s construction of General Relativity, a classical non-linear theory par-excellence, was very early on seen to raise the same questions for gravitating particles, and gravitating bodies more generally. Two questions arose immediately.

YYY

- (1) Does the vacuum theory (*i.e.* $R_{\mu\nu} = 0$) admit everywhere non-singular finite energy (*i.e.* asymptotically flat) solutions?

⁷The term *BIon* [49] should be carefully distinguished from *Soliton* which we take to be an *everywhere non-singular* solution of the classical equations of motion with finite total energy. The standard example of a soliton in relativistic field theory is the ‘t Hooft- Polyakov monopole [50, 51].

- (2) Do the equations of motion of gravitating bodies follow from the Einstein equations ($R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu}$) without needing to be postulated separately?

Question 1 was answered initially in the negative by Serini [17] and with increasing precision by Einstein and Pauli [18, 19] and Lichnerowicz [61].

Question 2 was answered in the positive by Einstein, Infeld and Hoffman [62]⁸.

The development of gauge theory and the discovery of non-abelian monopoles in Yang-Mills-Higgs theory [50, 51] sparked off renewed interest in classical models of particles or, in Coleman's happy phrase, classical lumps and their quantum descendants [63]. It was then natural to ask what were the analogues of these classical lumps in general relativity [64, 65]. Of course it is rather clear that given a stable finite energy soliton solution of a set of non-linear field equations in flat spacetime, such as a 't Hooft-Polyakov monopole, it should, when coupled to Einstein gravity, and if of a size large compared with its Schwarzschild radius, remain stable. However as its self-gravity becomes stronger it should undergo collapse to form a black hole [66, 67]. Among such possibilities are self-gravitating Q-balls [68, 69]. The scalar fields supporting Q-balls are necessarily time dependent, however the metric is time independent since it couples to the energy momentum tensor which is time independent for Q-matter. They are thus an interesting example of a solution of Einstein's equations for which the matter fields do not share all of the symmetries of the metric. Examples of this phenomenon occur in Einstein-Maxwell theory and involve time or space-dependent duality transformations [70]. In principle could occur in supergravity theories. However there is no example known to us which is asymptotically flat. In the present paper we shall always assume that *both* the matter and the metric are time independent.

The examples discussed in the previous paragraph, while of possible importance in nature, cannot be thought of as owing their origin to the properties of gravity, since they can be found in its absence. For the same reason, any topological characteristics relate to the topology of the space of field configurations, not to the topology of spacetime. Therefore, based on Lichnerowicz's theorem and its generalizations [71, 21] and Hawking's discovery of Black Hole Evaporation [72] it became clear that the only feasible candidate gravitational solitons in four dimensions were extreme black holes [64, 65] and this identification became even more attractive if they were embedded in supergravity theories since extreme black holes in Einstein-Maxwell theory are BPS. That is, they admit Killing spinors and fit into supermultiplets [73] and satisfy a Bogomolnyi bound [74, 75]. The absence of true, singularity free solitons in supergravity theories and an wide

⁸The basic point here is that the Bianchi identities imply (covariant) energy momentum conservation $T^{\mu\nu}{}_{;\nu} = 0$, which are the matter equations of motion. For electro-dynamics, however, the analogue of the Bianchi identities give a weaker result, current conservation $J^\mu{}_{;\mu} = 0$.

variety of related theories was confirmed in [20] using a generalization of the Smarr's formula for the total mass of a spacetime in terms of its charges and angular momentum and the area of any event horizon. In the absence of an horizon, Smarr's formula implies that the total energy M vanishes and hence by the positive energy theorem [76, 23], the solution is trivial.

Passing to five spacetime dimensions and imposing the vacuum equations, it is not difficult to see, using the positive action theorem [22, 23] that Lichnerowicz's theorem remains true, provided one insists that the solution be asymptotically Euclidean [77] but if one relaxes this assumption one obtains the Kaluza-Klein monopole [78, 79] whose supersymmetric credentials were established in [80]. It should be noted however, that, even in the complete absence of matter, supersymmetry will not prevent gravitational collapse and black hole formation, in either five [81], or nine [82] dimensional flat spacetime or of Kaluza-Klein monopoles [83] or their analogue in nine spacetime dimensions [84].

It would seem therefore that, just as in four spacetime dimensions, the natural candidates for solitons in five dimensional supergravity theories are extreme black holes admitting Killing spinors. However this is not true. The BPS fuzzball solutions of [10, 11, 13] are, as we shall show in detail in the body of this paper, complete and non-singular, with no horizons. Since a conventional Smarr formulae have been established for five dimensional black holes, at least in a special case [85], and a Lichnerowicz type theorem proved [86], the question naturally arises as to how the fuzzball solutions get round these No-Go Theorems. It is the purpose of this paper to answer this question. In brief, the answer is that if one includes both Chern-Simons terms, which were correctly included in [85] and takes into account the possibility that the four-dimensional spatial manifold may be topologically non-trivial, and in particular may have non-trivial second homology group, there is an extra bulk term in the Smarr formula which is sufficient to prevent the arguments in [20] being extended from four to five, or indeed higher, spacetime dimensions. To put this in slightly different terms: the combination of non-trivial topology and Chern-Simons interactions allows, in five spacetime dimensions, what is not possible in four spacetime dimensions; the concrete realisation of Wheeler and Misner's Geometrodynamics programme in which particles have their origin in field lines trapped in the topology of space [87].

3 Global structure

In this section we shall consider the problem in greater generality and treat D -dimensional spacetimes $\{M, g\}$ whose metrics are stationary (at least in some neighborhood of infinity) and so admit a Killing vector field $\mathbf{K} = K^\mu \frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial t}$ in adapted local coordinates $X^\mu = (t, x^i)$, $\mu =$

$0, 1, 2, \dots, D-1$. In these adapted fibre coordinates the metric takes the form:

$$ds_D^2 = g_{\mu\nu} dx^\mu dx^\nu = -\frac{1}{Z^2(x^j)}(dt + k_i(x^k)dx^i)^2 + \gamma_{ij}(x^k)dx^i dx^j. \quad (3.1)$$

Locally we may think of the spacetime manifold M as an \mathbb{R} -bundle over some D -dimensional space of orbits \mathcal{B} with coordinates x^i and we may think of the hypersurfaces $t = \text{constant}$ as a local section. The metric γ_{ij} is the projection of the spacetime metric $g_{\mu\nu}$ orthogonal to the fibres whose coordinate is t and the vector field, k_i , defines the horizontal, sometimes called in this context the Sagnac, connection. This vector field is also sometimes known as the angular momentum vector. The metric, γ_{ij} , is Riemannian as long as $\frac{\partial}{\partial t}$ is timelike, but is ill-defined at places where $\frac{\partial}{\partial t}$ becomes null. It should be distinguished from the metric induced on the local hypersurfaces $t = \text{constant}$:

$$g_{ij} = \gamma_{ij} - Z^{-2}k_i k_j. \quad (3.2)$$

If we change the section by means of the coordinate transformation

$$t \rightarrow \tilde{t} = t + f(x^i) \quad (3.3)$$

then the induced metric g_{ij} will change since

$$k_i \rightarrow \tilde{k}_i - \frac{\partial f}{\partial x^i}, \quad (3.4)$$

but the metric γ_{ij} orthogonal to the fibres will remain unchanged.

In what follows we will explore the global situation. We have, since $K^\mu = \delta_0^\mu$,

$$-g_{tt} = -g_{\mu\nu}K^\mu K^\nu = \frac{1}{Z^2}. \quad (3.5)$$

Thus, if $\frac{1}{Z^2} \geq 0$ everywhere, then the fibres are non-spacelike and become lightlike at points for which $\frac{1}{Z^2} = 0$. The “critical” hypersurface at which $\frac{1}{Z^2} = 0$, may be timelike or null. The latter represents a degenerate (or “extreme”) Killing horizon, *i.e.* a null hypersurface whose null generators coincide with the orbits of the Killing vector field \mathbf{K} . Since, locally, a future-directed causal (*i.e.* timelike or null) curve may only cross a null hypersurface in one direction, such a surface acts as a stationary one-way membrane.

Time-like critical hypersurfaces are less familiar. Such a surface is certainly not a one-way membrane since nothing locally prevents a future directed causal curve crossing a timelike hyperspace in either direction. If g_{tt} merely changed sign as one crossed a timelike hypersurface, then the hypersurface would locally bound a region, called an ergo-region, in which the Killing vector field becomes spacelike, and the surface on which $\frac{1}{Z^2} = 0$ would be an ergo-surface⁹.

⁹Often these surfaces have topology $\mathbb{R} \times S^{D-2}$ and are then called ergo-spheres.

However if g_{tt} is never positive, and so there is no ergo-region and no generally accepted term for the surface on which $\frac{1}{Z^2} = 0$. The occurrence of such surfaces, at which $\frac{1}{Z^2} = 0$ typically has a double zero, often arise in supersymmetric spacetimes for which \mathbf{K} has a spinorial square root, that is, there exists a spinor field ϵ for which

$$K^\mu = \bar{\epsilon} \gamma^\mu \epsilon. \quad (3.6)$$

One can show that the right-hand side of (3.6) is never spacelike. If $\frac{1}{Z^2}$ has a double zero, the normal to the hypersurface $\frac{1}{Z} = 0$ will be timelike if $g^{ij} \frac{\partial_i Z \partial_j Z}{Z^4}$ is positive there and null if $g^{ij} \frac{\partial_i Z \partial_j Z}{Z^4}$ vanishes there.

The section $t = \text{constant}$ has a normal $\partial_\mu t$

$$g^{tt} = g^{\mu\nu} \partial_\mu t \partial_\nu t = -(Z^2 - \gamma^{ij} k_i k_j) \quad (3.7)$$

and will be everywhere timelike, and the section everywhere spacelike if

$$(Z^2 - \gamma^{ij} k_i k_j) > 0. \quad (3.8)$$

Note that

$$\det g_{ij} = \frac{1}{Z^2} (\det \gamma_{ij}) (Z^2 - \gamma^{ij} k_i k_j) \quad (3.9)$$

and so a necessary condition that the metric induced on the section is positive definite is

$$\frac{1}{Z^2} (\det \gamma_{ij}) (Z^2 - \gamma^{ij} k_i k_j) > 0. \quad (3.10)$$

If conditions (3.8) and (3.10) hold globally then any future directed causal curve may cross a section once and only once and one may regard t as a *global time function*. The spacetime is then *stably causal* [93] and the sections $t = \text{constant}$ can be viewed as *global Cauchy surfaces*. If \mathcal{B} is the space of orbits, then the topology of the spacetime manifold will be a product $M \equiv \mathbb{R} \times \mathcal{B}$.

A simple example of the situation we are interested in is the product metric on $AdS_3 \times \frac{1}{4}S^2$ [10, 11, 13]

$$ds_5^2 = -\cosh^2 \xi d\tau^2 + d\xi^2 + \sinh^2 \xi d\varphi_1^2 + \frac{1}{4} (d\theta^2 + \sin^2 \theta d\varphi_2^2) \quad (3.11)$$

with the Killing vector

$$\mathbf{K} = \frac{\partial}{\partial \tau} - \frac{\partial}{\partial \varphi_1} + 2 \frac{\partial}{\partial \varphi_2} \quad (3.12)$$

for which

$$g(K, K) = -\cos^2 \theta. \quad (3.13)$$

One may think of the integral curves of \mathbf{K} as world-lines of non-space-like “observers” that are time-like everywhere except for $\theta = \frac{\pi}{2}$. The metric is clearly geodesically complete and non-singular and the Killing vector field \mathbf{K} regular and timelike everywhere except on the equator

of the S^2 , $\theta = \frac{\pi}{2}$. If we set $\theta = \frac{\pi}{2}$ in the metric (3.11) we get a regular four-dimensional metric of signature $-+++$, that is $\theta = \frac{\pi}{2}$ is a timelike hypersurface. While this example may seem a little contrived, we will see in Section 6.6.1 that it is, in fact, an archetypical model for the local description of critical surfaces in a fuzzball model.

4 Some Microstate geometries

4.1 The $\mathcal{N}=2$ supergravity theory

The simplest candidate microstate geometries have been constructed in $\mathcal{N}=2$, five-dimensional supergravity coupled to two vector multiplets. Including the gravi-photon there are three vector fields and two independent scalars which may conveniently be parametrized by the fields, X^I , $I = 1, 2, 3$ satisfying the constraint $X^1 X^2 X^3 = 1$. The bosonic action is

$$S = \int \sqrt{-g} d^5x \left(R - \frac{1}{2} Q_{IJ} F_{\mu\nu}^I F^{J\mu\nu} - Q_{IJ} \partial_\mu X^I \partial^\mu X^J - \frac{1}{24} C_{IJK} F_{\mu\nu}^I F_{\rho\sigma}^J A_\lambda^K \bar{\epsilon}^{\mu\nu\rho\sigma\lambda} \right), \quad (4.1)$$

with $I, J = 1, 2, 3$. The metric for the kinetic terms is

$$Q_{IJ} = \frac{1}{2} \text{diag} \left((X^1)^{-2}, (X^2)^{-2}, (X^3)^{-2} \right). \quad (4.2)$$

We are interested in five-dimensional stationary space-times, \mathcal{M}_5 , whose Lorentzian metric may be cast in the local form:

$$ds_5^2 = -Z^{-2} (dt + k)^2 + Z ds_4^2, \quad (4.3)$$

where, compared to (3.1), it is convenient to introduce the warp factor, Z , in front of the general metric, ds_4^2 on the four-dimensional base manifold, \mathcal{B} . The metric induced on the local hypersurfaces $t = \text{constant}$ is now given by:

$$ds_{\text{induced}}^2 = -Z^{-2} k^2 + Z ds_4^2. \quad (4.4)$$

The matter fields are assumed to be time-independent and therefore the Maxwell fields may be decomposed into electric and magnetic components:

$$A^I = -Z_I^{-1} (dt + k) + B^{(I)}, \quad (4.5)$$

where $B^{(I)}$ is a one-form on \mathcal{B} . It will prove convenient to define magnetic field strengths:

$$\Theta^{(I)} \equiv dB^{(I)}. \quad (4.6)$$

4.2 BPS solutions

The solutions of [88, 89, 10, 11, 13] are BPS, *i.e.* supersymmetric and they are obtained by requiring that they admit Killing spinor fields. Specifically, one seeks solutions that preserve exactly the same supersymmetries as a BPS black hole with the same electric charges. This leads to what later became known as the “floating brane Ansatz” [90] because the constituent charges are carried by branes and those branes must obey a ‘zero-force’ condition in a BPS solution. This means that the scalars and warp factors are related to the electric potentials via:

$$Z \equiv (Z_1 Z_2 Z_3)^{1/3}, \quad X^1 = \left(\frac{Z_2 Z_3}{Z_1^2} \right)^{1/3}, \quad X^2 = \left(\frac{Z_1 Z_3}{Z_2^2} \right)^{1/3}, \quad X^3 = \left(\frac{Z_1 Z_2}{Z_3^2} \right)^{1/3}. \quad (4.7)$$

The conformally rescaled base metric, ds_4^2 , is then required to be hyper-Kähler and supersymmetric configurations are obtained by solving the system of equations [26, 10, 11]:

$$\Theta^{(I)} = \star_4 \Theta^{(I)}, \quad (4.8)$$

$$\nabla^2 Z_I = \frac{1}{2} C_{IJK} \star_4 (\Theta^{(J)} \wedge \Theta^{(K)}), \quad (4.9)$$

$$dk + \star_4 dk = Z_I \Theta^{(I)}, \quad (4.10)$$

where \star_4 is the Hodge dual taken with respect to the four-dimensional metric, ds_4^2 , and the structure constants are given by $C_{IJK} \equiv |\epsilon_{IJK}|$. More generally, when the supergravity is coupled to more $\mathcal{N} = 2$ vector multiplets, these structure constants are precisely those that determine the structure of the vector multiplet sector and its scalar coset.

For the metric (4.3) to be asymptotically flat and the vector kinetic term in (4.1) to be well-behaved at infinity one usually requires that Z_I goes to a non-zero constant at infinity. By rescaling coordinates and fields one can, without loss of generality, take

$$Z_I \rightarrow 1 \quad (4.11)$$

at infinity. This will then give the vector kinetic term its canonical normalization at infinity. As we will see in Section 6.6.1, if one wants different asymptotics then one does not necessarily impose (4.11).

5 Smarr Formula in 4 + 1 spacetime dimensions

In many circumstances, Smarr’s formula enables one to relate the mass of a solution to properties on interior boundaries. Moreover, if there are no such boundaries because the solution is smooth and horizonless, one typically finds that the mass must be zero. Thus Smarr’s formula lies at the heart of the belief that there are “No solitons without horizons.” We will show how solutions

arising from the action (4.1) avoid this conclusion precisely because of the Chern-Simons term. Interestingly enough, the role of Chern-Simons terms has been carefully analysed in the context of horizon topologies [91, 92] but the consequences of topological Chern-Simons contributions in the bulk space-time do not appear to have been considered to date.

5.1 Equations of Motion

The Einstein equations coming from (4.1) are:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = Q_{IJ} \left[F_{\mu\rho}^I F_{\nu}^{J\rho} - \frac{1}{4}g_{\mu\nu} F_{\rho\sigma}^I F^{J\rho\sigma} + \partial_\mu X^I \partial_\nu X^J - \frac{1}{2}g_{\mu\nu} g^{\rho\sigma} \partial_\rho X^I \partial_\sigma X^J \right]. \quad (5.1)$$

Taking traces and rearranging gives the equation:

$$R_{\mu\nu} = Q_{IJ} \left[F_{\mu\rho}^I F_{\nu}^{J\rho} - \frac{1}{6}g_{\mu\nu} F_{\rho\sigma}^I F^{J\rho\sigma} + \partial_\mu X^I \partial_\nu X^J \right]. \quad (5.2)$$

The Maxwell equations coming from (4.1) are:

$$\nabla_\rho (Q_{IJ} F^{J\rho}{}_\mu) = J_{I\mu}^{CS}, \quad (5.3)$$

where the Chern-Simons currents are given by:

$$J_{I\mu}^{CS} \equiv \frac{1}{16} C_{IJK} \epsilon_{\mu\alpha\beta\gamma\delta} F^{J\alpha\beta} F^{K\gamma\delta}. \quad (5.4)$$

Define dual 3-forms, G , by

$$G_{I\rho\mu\nu} \equiv \frac{1}{2} Q_{IJ} F^{J\alpha\beta} \epsilon_{\alpha\beta\rho\mu\nu} \quad (5.5)$$

and introduce the inverse, Q^{IJ} of Q_{IJ} :

$$Q^{IJ} Q_{JK} = \delta_K^I. \quad (5.6)$$

It follows from the Bianchi identities for $F_{\mu\nu}^J$ that G_J satisfies:

$$\nabla_\rho (Q^{IJ} G_J^{\mu\nu\rho}) = 0, \quad (5.7)$$

Similarly, from the equations of motion (5.3) for $F_{\mu\nu}^J$ one has

$$\nabla_{[\lambda} G_{J|\rho\mu\nu]} = +\frac{3}{8} C_{IJK} F_{[\lambda\rho}^J F_{\mu\nu]}^K \Leftrightarrow dG_I = +\frac{1}{4} C_{IJK} F^J \wedge F^K. \quad (5.8)$$

where $|J|$ means that the index J is not involved in the skew-symmetrization bracket $[\dots]$.

One can easily verify that

$$Q^{IJ} G_{I\mu\rho\sigma} G_J^{\nu\rho\sigma} = Q_{IJ} (2 F_{\mu\rho}^I F^{J\nu\rho} - \delta_\mu^\nu F_{\rho\sigma}^I F^{J\rho\sigma}) \quad (5.9)$$

and so we may rewrite the Einstein equation (5.2) as

$$R_{\mu\nu} = Q_{IJ} \left[\frac{2}{3} F_{\mu\rho}^I F_{\nu}^{J\rho} + \partial_\mu X^I \partial_\nu X^J \right] + \frac{1}{6} Q^{IJ} G_{I\mu\rho\sigma} G_J^{\nu\rho\sigma}. \quad (5.10)$$

5.2 Invariances

As remarked upon above, we shall make the assumption that the matter fields share the symmetry of the metric. In particular we assume that they are invariant under diffeomorphisms generated by the Killing vector, K^μ :

$$\mathcal{L}_K F^I = 0, \quad \mathcal{L}_K G_I = 0, \quad \mathcal{L}_K X^I = 0, \quad (5.11)$$

where \mathcal{L}_K denotes the Lie derivative. A formula of Cartan states that for a p -form, α , one has

$$\mathcal{L}_K \alpha = d(i_K(\alpha)) + i_K(d\alpha). \quad (5.12)$$

Taking $\alpha = F^I$ we have, locally,

$$K^\rho F^I_{\rho\mu} = \partial_\mu \lambda^I, \quad (5.13)$$

for some functions λ^I .

If the space-time manifold were not simply connected one could, in principle, encounter jumps in value of λ^I if one were to integrate (5.13) around a closed curve. To avoid this issue we shall, from now on, assume that our space-time manifold, \mathcal{M}_5 is simply connected. With this assumption, the arbitrary constants in the definitions of the functions, λ^I , may be fixed by requiring that the λ^I vanish at infinity. As we will see, in Section 8.1, this choice of boundary condition will crucially affect the details of the Smarr formula. Physically, the functions, λ^I , are magnetostatic potentials of the 3-forms, G_I , or, equivalently, electrostatic potentials of the 2-forms, F^I .

Taking $\alpha = G_I$ we have

$$\begin{aligned} d(i_K(G_I)) &= -i_K(dG_I) = -\frac{1}{4} C_{ILM} i_K(F^L \wedge F^M) = -\frac{1}{2} C_{ILM} d\lambda^L \wedge F^M \\ &= -\frac{1}{2} C_{ILM} d(\lambda^L F^M) \end{aligned} \quad (5.14)$$

where we have used (5.8) and (5.13). While the assumption of simple connectivity is a weak one because we could pass to a covering space, we cannot assume that the $H^2(\mathcal{M}_5)$ is trivial. Indeed, this is the crucial issue that makes solitons possible. Thus we deduce that

$$K^\rho G_{I\rho\mu\nu} = \partial_\mu \Lambda_{I\nu} - \partial_\nu \Lambda_{I\mu} - \frac{1}{2} C_{IJK} \lambda^J F_{\mu\nu}^K + H_{I\mu\nu}, \quad (5.15)$$

where Λ_I are globally defined one-forms and H_I are closed but not exact two forms. That is, we cannot write $H_I = d\nu_I$ where ν_I are globally well-defined one-forms.

Using (5.13) and (5.15) we see that

$$K^\mu (Q_{IJ} F^I_{\mu\rho} F^J_{\nu}{}^\rho) = -\nabla_\rho (Q_{IJ} \lambda^I F^{J\rho\nu}) + \frac{1}{16} C_{IJK} \epsilon^{\nu\alpha\beta\gamma\delta} \lambda^I F^J_{\alpha\beta} F^K_{\gamma\delta} \quad (5.16)$$

$$\begin{aligned} K^\mu (Q^{IJ} G_{I\mu\rho\sigma} G_J{}^{\nu\rho\sigma}) &= -2\nabla_\rho (Q^{IJ} \Lambda_{I\sigma} G_J{}^{\rho\nu\sigma}) - \frac{1}{4} C_{IJK} \epsilon^{\nu\alpha\beta\gamma\delta} \lambda^I F^J_{\alpha\beta} F^K_{\gamma\delta} \\ &\quad + Q^{IJ} H_I{}^{\rho\sigma} G_{J\rho\sigma\nu} \end{aligned} \quad (5.17)$$

and hence, Einstein's equations (5.10) become:

$$K^\mu R_{\mu\nu} = -\frac{1}{3} \nabla^\mu \left[2 Q_{IJ} \lambda^I F^J{}_{\mu\nu} + Q^{IJ} \Lambda_I{}^\sigma G_{J\mu\nu\sigma} \right] + \frac{1}{6} Q^{IJ} H_I^{\rho\sigma} G_{J\rho\sigma\nu}, \quad (5.18)$$

where we have used $K^\mu \partial_\mu X^I = \mathcal{L}_K X^I = 0$. Note that the $\lambda(*F \wedge F)$ terms have canceled in (5.18). The last term in (5.18) may be expressed as

$$\frac{1}{6} Q^{IJ} H_I^{\rho\sigma} G_{J\rho\sigma\nu} = \frac{1}{12} \epsilon_{\alpha\beta\rho\sigma\nu} F^{I\alpha\beta} H_I^{\rho\sigma}. \quad (5.19)$$

A similar result was obtained in [85], where it was also noted that precisely in five dimensions the $\lambda(*F \wedge F)$ cancel out in the expression (5.18). However, they assumed the global existence of the vector potentials, A^I . In the “no-go” theorem of [86] there were no Chern-Simons terms but in their equations (37) and (38) they assumed the existence of global potentials, Ψ and Φ , and a crucial term, analogous to (5.19) is missing in their equation (48).

5.3 Mass and Charge

5.3.1 Expansions at infinity

To get the correctly normalized asymptotic charges for an asymptotically flat metric in a D -dimensional space-time one should start from the canonically normalized action:

$$S = \int d^D x \sqrt{-g} \left(\frac{R}{16\pi G_D} + \mathcal{L}_{\text{matter}} \right), \quad (5.20)$$

where G_D is the Newton constant. The Einstein equations are, as usual,

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G_D T_{\mu\nu}, \quad (5.21)$$

where $T_{\mu\nu}$ is the canonically normalized energy-momentum tensor. The Einstein equations may be rewritten as

$$R_{\mu\nu} = 8\pi G_D \left(T_{\mu\nu} - \frac{1}{(D-2)} T g_{\mu\nu} \right), \quad (5.22)$$

where T is the trace of $T_{\mu\nu}$.

If one linearizes around a flat metric one can then define the momentum and angular momentum of the configuration by integrating over a space-like hyper surface, Σ :

$$P^\mu = \int_\Sigma d^{D-1} x T^{\mu 0}, \quad J^{\mu\nu} = \int_\Sigma d^{D-1} x (x^\mu T^{\nu 0} - x^\nu T^{\mu 0}). \quad (5.23)$$

One can then use the linearized Einstein equations to show that in a rest frame one has [95, 96, 97, 98]:

$$g_{00} = -1 + \frac{16\pi G_D}{(D-2)A_{D-2}} \frac{M}{\rho^{D-3}} + \dots, \quad (5.24)$$

$$g_{ij} = 1 + \frac{16\pi G_D}{(D-2)(D-3)A_{D-2}} \frac{M}{\rho^{D-3}} + \dots, \quad (5.25)$$

$$g_{0i} = \frac{16\pi G_D}{A_{D-2}} \frac{x^j J^{ji}}{\rho^{D-1}} + \dots, \quad (5.26)$$

where ρ is the radial coordinate and A_{D-2} is the volume of a unit $(D-2)$ sphere. In particular, one has $A_3 = 2\pi^2$.

Note that in the linearized system it follows from (5.22) that

$$R_{00} \approx 8\pi G_D \left(T_{00} - \frac{1}{(D-2)} g^{00} T_{00} g_{00} \right) = 8\pi G_D \frac{(D-3)}{(D-2)} T_{00}. \quad (5.27)$$

For the Maxwell action

$$S_{\text{Maxwell}} = \int d^D x \sqrt{-g} \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right), \quad (5.28)$$

the asymptotic electric charge is given by the expansion:

$$F_{0\rho} = (D-3) \frac{Q}{\rho^{D-2}}. \quad (5.29)$$

Note that the more standard normalization of charge that is adapted to Gaussian integrals would have a different overall factor involving $A_{(D-2)}^{-1}$ but here we have chosen to normalize the $U(1)$ charges so as to be consistent with the usual literature on bubbled geometries (see, for example, [13]).

More generally, the expansions (5.24)–(5.26) and (5.29) may be used to define asymptotic charges of a generic, asymptotically-flat metric.

5.3.2 Normalizing the Komar Integrals

If the metric has Killing vectors then they can be used to define a globally conserved quantity via a Komar integral. Indeed, if K is a time-like Killing vector then the following Komar integral defines a conserved mass:

$$\int_{S^{D-2}} *dK = \int_{S^{D-2}} (\partial_\mu K_\nu - \partial_\nu K_\mu) d\Sigma^{\mu\nu}. \quad (5.30)$$

where S^{D-2} is a closed, space-like surface at infinity. If the configuration is smooth on a space-like hypersurface, Σ , then one can write

$$\int_{S^{D-2}} *dK = \int_\Sigma d * dK = -2 \int_\Sigma *(K^\mu R_{\mu\nu} dx^\nu). \quad (5.31)$$

where we have used the fact that for a Killing vector $\nabla^2 K^\mu = -R_{\mu\nu}K^\nu$.

To normalize the Komar mass one can consider, once again, a linearization and use (5.27) to obtain:

$$M = -\frac{1}{16\pi G_D} \frac{(D-2)}{(D-3)} \int_{S^{D-2}} *dK = -\frac{1}{16\pi G_D} \frac{(D-2)}{(D-3)} \int_{S^{D-2}} (\partial_\mu K_\nu - \partial_\nu K_\mu) d\Sigma^{\mu\nu}. \quad (5.32)$$

One may check this normalization against that of (5.24) but observing that to leading order at infinity, $K = g_{00}dt$ and hence

$$*dK \approx -(\partial_\rho g_{00}) * (dt \wedge d\rho) \approx -\frac{16\pi G_D (D-3)}{(D-2) A_{D-2}} \frac{M}{\rho^{D-2}} d\text{vol}(S^{D-2}), \quad (5.33)$$

which is consistent with (5.32).

5.3.3 Komar Integrals in five dimensional supergravity

We now specialize to our five-dimensional theory and take as our starting point the Komar integral for the ADM mass:

$$\frac{16\pi G_5}{3} M = -\frac{1}{2} \int_\infty (\partial_\mu K_\nu - \partial_\nu K_\mu) d\Sigma^{\mu\nu}, \quad (5.34)$$

where the integral is taken over an S^3 at spatial infinity and $d\Sigma^{\mu\nu}$ is the volume form on this sphere.

Once again we suppose that there is a Cauchy surface, Σ , whose boundary at infinity is the S^3 above but now we will allow it to also have interior boundaries, $\partial\Sigma_{int}$. Then we find:

$$\frac{16\pi G_5}{3} M = \int_\Sigma R_{\mu\nu} K^\mu d\Sigma^\nu + \frac{1}{2} \int_{\partial\Sigma_{int}} (\partial_\mu K_\nu - \partial_\nu K_\mu) d\Sigma^{\mu\nu} \quad (5.35)$$

$$= \int_\Sigma \left[\frac{1}{6} Q^{IJ} H_{I\rho\sigma} G_J^{\rho\sigma\nu} - \frac{1}{3} \nabla_\mu \left(2 Q_{IJ} \lambda^I F^{J\mu\nu} + Q^{IJ} \Lambda_{I\sigma} G_J^{\sigma\mu\nu} \right) \right] d\Sigma_\nu \quad (5.36)$$

$$+ \frac{1}{2} \int_{\partial\Sigma_{int}} (\partial_\mu K_\nu - \partial_\nu K_\mu) d\Sigma^{\mu\nu}, \quad (5.37)$$

where we have used (5.18). If we assume that the second term on the right-hand side of (5.36) falls off sufficiently fast at infinity then we may write the mass as

$$\begin{aligned} \frac{16\pi G_5}{3} M &= \frac{1}{6} \int_\Sigma \left[Q^{IJ} H_{I\rho\sigma} G_J^{\rho\sigma\nu} \right] d\Sigma_\nu \\ &+ \int_{\partial\Sigma_{int}} \left[-\frac{1}{3} \left(2 Q_{IJ} \lambda^I F^{J\mu\nu} + Q^{IJ} \Lambda_{I\sigma} G_J^{\sigma\mu\nu} \right) + \frac{1}{2} (\partial_\mu K_\nu - \partial_\nu K_\mu) \right] d\Sigma^{\mu\nu}. \end{aligned} \quad (5.38)$$

In the standard applications of the Smarr formula (see, for example, [85, 86]), the two-form, H_I , is assumed to be zero and the interior boundaries are horizons. Thus the boundary terms relate the ADM mass to horizon areas, charges and angular momenta. If there are no horizons and one assumes that $H_I = 0$ then the ADM mass vanishes. However, using (5.35), we have

$$\frac{16\pi G_5}{3} M = \int_{\Sigma} R_{00} d^4x. \quad (5.39)$$

From (5.10) we see that $R_{00} \geq 0$ and vanishes if and only if $F_{0j}^I = 0$, $\partial_t X^I = 0$ and $G_{0ij} = 0$. Since G_I is the dual of F^I , this means that $F_{ij}^I = 0$ and hence F^I and G_I vanish identically. The scalars therefore have no source and if one parameterizes them with $X^1 = e^{\phi_1 + \phi_2}$, $X^2 = e^{\phi_1 - \phi_2}$ and $X^3 = e^{-2\phi_1}$ then the ϕ_a must be harmonic. Since $\partial_t \phi_a = 0$ there are no non-trivial solutions that are smooth and fall off at infinity. Hence we must have $\phi_a = 0$ and $X^I = 1$ everywhere. Thus the complete solution would necessarily be trivial. This result is consistent with [86], which analyses a simpler theory with a single 3-form with no Chern-Simons term and makes the assumption that the potentials are globally well-defined.

On the other hand, if H_I is not zero and there are no inner boundaries, we conclude that M is not only positive but must be given by:

$$M = \frac{1}{32\pi G_5} \int_{\Sigma} \left[Q^{IJ} H_{I\rho\sigma} G_J^{\rho\sigma\nu} \right] d\Sigma_{\nu}. \quad (5.40)$$

In the corresponding situation in four space-time dimensions there is no analogue of the H_I and one concludes [20] that for ungauged supergravity theories in general there are no soliton solutions unless horizons are present. In five dimensions however, we have seen that provided our space-time is topologically non-trivial, with non-vanishing $H^2(\mathcal{M}_5)$, there is no obstacle to regular solitons states without horizons.

6 A class of examples

6.1 The hyper-Kähler base

Perhaps the simplest hyper-Kähler metrics are those based upon a $U(1)$ fibration over a flat \mathbb{R}^3 :

$$ds_4^2 = h_{ij} dx^i dx^j = \frac{1}{V(y^a)} (d\psi + A_a(y^a) dy^a)^2 + V(y^a) dy^a dy^a, \quad (6.1)$$

where $a = 1, 2, 3$ and

$$\vec{\nabla} \times \vec{A} = \vec{\nabla} V \quad (6.2)$$

where $\vec{\nabla}$ denotes the standard gradient operator, $\frac{\partial}{\partial y^a}$, on flat, Euclidean \mathbb{R}^3 . Note that (6.2) implies that V is a harmonic function on \mathbb{R}^3 . The circle action $\mathbf{L} = \frac{\partial}{\partial \psi}$ generated is tri-holomorphic

and the triple of functions, y^a , are the moment maps for the tri-holomorphic circle action generated by \mathbf{L} .

The harmonic functions, V , that we will consider take the form

$$V = \varepsilon_0 + \sum_{j=1}^N \frac{q_j}{|\vec{y} - \vec{y}^{(j)}|} \quad (6.3)$$

where ε_0 may be zero or one, and the q_j may be plus or minus one. Thus, in general V will not be positive, and the metric signature will change from $+4$ to -4 when V changes sign.

However we can always find solutions in which ZV is globally positive:

$$ZV > 0, \quad (6.4)$$

and smooth, except possibly at the points $\vec{y}^{(j)}$, at which Z is finite and hence, near $\vec{y}^{(j)}$, one has

$$ZV \sim \frac{z_j q_j}{|\vec{y} - \vec{y}^{(j)}|}, \quad (6.5)$$

for some constants, z_j . If $|q_j| = 1$, then the apparent singularity of the metric at $\vec{y}^{(j)}$ may be eliminated by making the identification

$$0 \leq \psi \leq 4\pi. \quad (6.6)$$

and changing variables so that $|\vec{y} - \vec{y}^{(j)}| = \frac{1}{4}R^2$. It follows that the 4-metric

$$Zds_4^2 = \frac{Z}{V}(d\psi + \vec{A} \cdot d\vec{y})^2 + ZV d\vec{y} \cdot d\vec{y}, \quad (6.7)$$

is smooth and Riemannian in neighborhoods of the points $\vec{y}^{(j)}$, provided that $|q_j| = 1$. If $|q_j| \in \mathbb{Z}_+$ then the manifold has a simple $\mathbb{Z}_{|q_j|}$ orbifold singularity at $\vec{y}^{(j)}$. Indeed, away from the locus $V = 0$, the metric (6.7) is non-singular and Riemannian. However, the metric (6.7) in the direction of the circle fibre is singular at $V = 0$. We will refer to the surfaces defined by $V = 0$ as *critical surfaces* and in Section 6.3.1, we will describe how this singularity at $V = 0$ is cancelled in the full five-dimensional metric, (4.3), by terms coming from the angular momentum vector, k .

The behaviour of the metric at infinity is determined by the asymptotic behaviour of V in (6.3) and so we define the parameter, q_0 , by

$$q_0 \equiv \sum_{j=1}^N q_j. \quad (6.8)$$

If $\varepsilon_0 \neq 0$ then (6.1) is asymptotic to the flat metric on $\mathbb{R}^3 \times S^1$ but if $\varepsilon_0 = 0$ then the metric (6.1) is asymptotic to flat \mathbb{R}^4 if and only if $q_0 = +1$.

6.2 The complete solution

We now describe the ψ -independent solutions [89, 94, 10, 11, 13] of the BPS equations (4.8)–(4.10) for the base metric (6.1). Introduce a set of frames

$$\hat{e}^1 = V^{-\frac{1}{2}}(d\psi + A), \quad \hat{e}^{a+1} = V^{\frac{1}{2}}dy^a, \quad a = 1, 2, 3, \quad (6.9)$$

and two associated sets of two-forms:

$$\Omega_{\pm}^{(a)} \equiv \hat{e}^1 \wedge \hat{e}^{a+1} \pm \frac{1}{2} \epsilon_{abc} \hat{e}^{b+1} \wedge \hat{e}^{c+1}, \quad a = 1, 2, 3. \quad (6.10)$$

The two-forms, $\Omega_{-}^{(a)}$, are anti-self-dual, harmonic and non-normalizable and they define the hyper-Kähler structure on the base. The forms, $\Omega_{+}^{(a)}$, are self-dual and can be used to construct harmonic fluxes that are dual to the two-cycles. The Maxwell fields, $\Theta^{(I)}$, are required by (4.8) to self-dual and have the form

$$\Theta^{(I)} \equiv \sum_{a=1}^3 (\partial_a (V^{-1} K^I)) \Omega_{+}^{(a)}. \quad (6.11)$$

where the K^I are harmonic in \mathbb{R}^3 , *i.e.* $\nabla^2 K^I = 0$. It is straightforward to find a *local* potential such that $\Theta^{(I)} = dB^I$:

$$B^I \equiv V^{-1} K^I (d\psi + A) + \vec{\xi}^I \cdot d\vec{y}, \quad (6.12)$$

where

$$\vec{\nabla} \times \vec{\xi}^I = -\vec{\nabla} K^I. \quad (6.13)$$

The solution to (4.9) for Z_I is

$$Z_I = \frac{1}{2} C_{IJK} V^{-1} K^J K^K + L_I, \quad (6.14)$$

where $C_{IJK} \equiv |\epsilon_{IJK}|$ and the L_I are three more independent harmonic functions. If one writes the angular-momentum vector, k , as:

$$k = \mu (d\psi + A) + \omega, \quad (6.15)$$

then the solution to (4.10) is given by:

$$\mu = \frac{1}{6} C_{IJK} \frac{K^I K^J K^K}{V^2} + \frac{1}{2V} K^I L_I + M, \quad (6.16)$$

where M is yet another harmonic function¹⁰ on \mathbb{R}^3 . One also has

$$\vec{\nabla} \times \vec{\omega} = V \vec{\nabla} M - M \vec{\nabla} V + \frac{1}{2} (K^I \vec{\nabla} L_I - L_I \vec{\nabla} K^I). \quad (6.17)$$

¹⁰In this section we are using the standard notation for describing these solutions and one should note, in particular, that M is *not* the mass of the solution.

The obvious multi-centre solution has:

$$V = \varepsilon_0 + \sum_{j=1}^N \frac{q_j}{r_j}, \quad K^I = k_0^I + \sum_{j=1}^N \frac{k_j^I}{r_j}, \quad (6.18)$$

$$L^I = \ell_0^I + \sum_{j=1}^N \frac{\ell_j^I}{r_j}, \quad M = m_0 + \sum_{j=1}^N \frac{m_j}{r_j}, \quad (6.19)$$

where $r_j \equiv |\vec{y} - \vec{y}^{(j)}|$. Regularity (finiteness) of the Z_I and μ as $r_j \rightarrow 0$ requires

$$\ell_j^I = -\frac{1}{2} C_{IJK} \frac{k_j^J k_j^K}{q_j}, \quad j = 1, \dots, N; \quad (6.20)$$

$$m_j = \frac{1}{12} C_{IJK} \frac{k_j^I k_j^J k_j^K}{q_j^2} = \frac{1}{2} \frac{k_j^1 k_j^2 k_j^3}{q_j^2}, \quad j = 1, \dots, N. \quad (6.21)$$

In order to obtain solutions that are asymptotic to five-dimensional Minkowski space, $\mathbb{R}^{4,1}$, one must take $\varepsilon_0 = 0$, $q_0 = 1$ and $k_0^I = 0$ in (6.18), where q_0 is defined in (6.8). Moreover, μ must vanish at infinity, and this fixes m_0 . As we noted earlier, we also take $Z_I \rightarrow 1$ as $r \rightarrow \infty$.

Hence, the solutions that are asymptotic to five-dimensional Minkowski space have:

$$\varepsilon_0 = 0, \quad q_0 = 1, \quad k_0^I = 0, \quad \ell_0^I = 1, \quad m_0 = -\frac{1}{2} q_0^{-1} \sum_{j=1}^N \sum_{I=1}^3 k_j^I. \quad (6.22)$$

The important physical point is that once the q_j and k_j^I have been chosen, the remaining parameters are basically fixed by (6.21). The free parameters, q_j and k_j^I , determine the magnetic fluxes on the 2-cycles.

While the functions Z_I and μ as well as the magnetic fields (6.11) are all singular on the critical surfaces, defined by $V = 0$, the remarkable thing is that the complete five-dimensional metric (4.3) and the complete electromagnetic field (4.5) are smooth in a neighbourhood of these critical surfaces. Indeed from the explicit expressions (6.14) and (6.16) one can verify that the terms involving negative powers of V cancel out in the complete Maxwell field and metric.

6.3 Regularity and topology

The parametrization of the scalar fields in (4.7) requires that all the functions, Z_I , have the same sign and positive definiteness of the spatial part of the metric (4.3) along the \mathbb{R}^3 directions of (6.1) requires that $ZV > 0$. Thus we must have

$$Z_I V > 0, \quad I = 1, 2, 3, \quad (6.23)$$

globally. Constant time slices of the five-dimensional metric gives the four-dimensional metrics

$$d\hat{s}_4^2 = -Z^{-2} (\mu(d\psi + A) + \omega)^2 + ZV^{-1}(d\psi + A)^2 + ZV(dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2) \quad (6.24)$$

$$= \frac{\mathcal{Q}}{Z^2 V^2} \left(d\psi + A - \frac{\mu V^2}{\mathcal{Q}} \omega \right)^2 + ZV \left(r^2 \sin^2 \theta d\phi^2 - \frac{\omega^2}{\mathcal{Q}} \right) + ZV(dr^2 + r^2 d\theta^2) \quad (6.25)$$

where

$$Z \equiv (Z_1 Z_2 Z_3)^{1/3}, \quad \mathcal{Q} \equiv Z_1 Z_2 Z_3 V - \mu^2 V^2. \quad (6.26)$$

It is also useful to recall that the complete gauge potentials are given by (4.5).

6.3.1 Regularity at critical surfaces

First observe that at $V = 0$, the component pieces of the vector potentials, $Z_I^{-1}(dt + k)$ and $B^{(I)}$, have singular components along $d\psi$. However, as $V \rightarrow 0$, the diverging terms in the complete vector potentials behave as:

$$A^{(I)} \sim \left(\frac{K^I}{V} - \frac{\mu}{Z_I} \right) (d\psi + A) \sim \left(\frac{K^I}{V} - \frac{K^1 K^2 K^3}{\frac{1}{2} V C_{IJK} K^J K^K} \right) (d\psi + A) \sim 0. \quad (6.27)$$

Thus $A^{(I)}$ is, in fact, regular on the critical ($V = 0$) surfaces.

Similarly, the component parts of the five-dimensional metric, (4.3), are also singular a $V = 0$. The $(d\psi + A)$ components of the angular momentum vector, k , diverges as V^{-2} while the coefficient of $(dt + k)^2$ is Z^{-2} , which vanishes as V^2 . The cross terms, $dt k$, remains finite at $V = 0$ and the only danger lies in the $(d\psi + A)^2$ terms, and since ZV is finite and positive, the scale of this circle is determined by \mathcal{Q} in (6.25) and (6.26). Indeed, it appears from (6.26) that this could diverge as V^{-2} . However, there is once again a remarkable cancellation of all the negative powers of V in \mathcal{Q} and a tedious computation yields the remarkable result:

$$\begin{aligned} \mathcal{Q} = & -M^2 V^2 - \frac{1}{3} M C_{IJK} K^I K^J K^K - M V K^I L_I - \frac{1}{4} (K^I L_I)^2 \\ & + \frac{1}{6} V C^{IJK} L_I L_J L_K + \frac{1}{4} C^{IJK} C_{IMN} L_J L_K K^M K^N \end{aligned} \quad (6.28)$$

with $C^{IJK} \equiv C_{IJK} = |\epsilon_{IJK}|$. This is obviously finite on $V = 0$ surfaces. The quantity, \mathcal{Q} , is, in fact, the $E_{7(7)}$ quartic invariant written in terms of the eight functions V, K^I, L_I and $2M$. This quantity is thus duality invariant and for black-hole solutions it determines the horizon area of the four-dimensional black hole obtained by reducing on the ψ -fibre.

6.3.2 Topology, fluxes and closed time-like curves

The topology of the five-dimensional metric is determined by that of the four-dimensional base, (6.1). Suppose, for the moment, that $V > 0$. Then representative cycles, Δ_{ij} , of the non-trivial

homology classes can be defined by the ψ -fibre over any simple curve between $\vec{y}^{(i)}$ and $\vec{y}^{(j)}$. The fact that V is singular at these points means that the fibre pinches off thereby defining a compact cycle. Let $\widehat{\Delta}_{ij}$ be the cycle Δ_{ij} with $\vec{y}^{(i)}$ and $\vec{y}^{(j)}$ excised. Since $\Theta^{(I)}$ is regular at $\vec{y}^{(i)}$ and $\vec{y}^{(j)}$, the flux through $\widehat{\Delta}_{ij}$ is the same as the flux through Δ_{ij} . Moreover, the vector potential, B^I can be globally defined on $\widehat{\Delta}_{ij}$ because the Dirac strings can be run out through the excised points. Thus, the magnetic flux through Δ_{ij} is given by:

$$\Pi_{ij}^{(I)} \equiv \frac{1}{4\pi} \int_{\Delta_{ij}} \Theta^{(I)} = \frac{1}{4\pi} \int_{\widehat{\Delta}_{ij}} \Theta^{(I)} = \frac{1}{4\pi} \int_{\partial \widehat{\Delta}_{ij}} B^I \quad (6.29)$$

$$= \frac{1}{4\pi} \int_0^{4\pi} d\psi (B^I|_{\vec{y}^{(j)}} - B^I|_{\vec{y}^{(i)}}). \quad (6.30)$$

Now arrange cylindrical polar coordinates, (ρ, ϕ, z) , so that $\vec{y}^{(i)}$ and $\vec{y}^{(j)}$ lie on the z -axis at $z = a_i$ and $z = a_j$. Near the point $\vec{y}^{(i)}$ one can easily check that the vector potential, B^I , in (6.12) is given by

$$B^I \sim \frac{k_i^I}{q_i} \left(d\psi + q_i \left(\frac{(z - a_i)}{|\vec{y} - \vec{y}^{(i)}|} + c_i \right) d\phi \right) - k_i^I \left(\frac{(z - a_i)}{|\vec{y} - \vec{y}^{(i)}|} + c_i \right) d\phi \sim \frac{k_i^I}{q_i} d\psi. \quad (6.31)$$

Therefore, the flux is given by

$$\Pi_{ij}^{(I)} = \left(\frac{k_j^I}{q_j} - \frac{k_i^I}{q_i} \right). \quad (6.32)$$

This description of the homology cycles and fluxes obviously generalizes to cycles that do not cross critical ($V = 0$) surfaces in ambipolar base geometries but the foregoing argument manifestly breaks down if one crosses a critical surface because $\Theta^{(I)}$ is singular.

However, given the regularity of the complete Maxwell field and five-dimensional metric, one should, of course define the magnetic fluxes by integrating $F^{(I)}$ over the same pieces of the geometry. Since the full vector potentials are smooth, except at the points $\vec{y}^{(i)}$ and $\vec{y}^{(j)}$ one obtains, once again:

$$\widetilde{\Pi}_{ij}^{(I)} \equiv \frac{1}{4\pi} \int_{\Delta_{ij}} F^{(I)} = \frac{1}{4\pi} \int_0^{4\pi} d\psi (A^I|_{\vec{y}^{(j)}} - A^I|_{\vec{y}^{(i)}}). \quad (6.33)$$

Using (4.5) and (6.15) and recalling that the ℓ_i^I and m_i were chosen in (6.21) so as to make the Z_I and μ finite at the $\vec{y}^{(i)}$, one finds

$$\widetilde{\Pi}_{ij}^{(I)} = \Pi_{ij}^{(I)} - \left(\frac{\mu(\vec{y}^{(j)})}{Z_I(\vec{y}^{(j)})} - \frac{\mu(\vec{y}^{(i)})}{Z_I(\vec{y}^{(i)})} \right). \quad (6.34)$$

The fact that the ψ -fibre pinches off at the $\vec{y}^{(i)}$ the metric (6.7) means that the ψ -circles would become closed time-like curves in the neighborhood of the $\vec{y}^{(i)}$ because $\mu(\vec{y}^{(i)})$ is generically finite in (6.25). Thus we must additionally impose the condition

$$\mu(\vec{y}^{(j)}) = 0, \quad j = 1, \dots, N. \quad (6.35)$$

We will discuss this more below, but here we note the important consequence of this for the topology.

A finite value of $\mu(\vec{y}^{(i)})$ would, of course, have opened up the collapsing curves that define the homology cycles. Requiring (6.35) means that in the full Lorentzian geometry, the ψ -circles are pinching off in exactly the manner that was naively suggested by the analysis in the spatial base and thus the naive topology is indeed the exact topology of the full, five-dimensional geometry. Furthermore, requiring (6.35) also means that

$$\tilde{\Pi}_{ij}^{(I)} \equiv \frac{1}{4\pi} \int_{\Delta_{ij}} F^{(I)} = \Pi_{ij}^{(I)} = \left(\frac{k_j^I}{q_j} - \frac{k_i^I}{q_i} \right). \quad (6.36)$$

Thus the cohomological fluxes are indeed the naive magnetic fluxes computed on the base and the result remains true even if the cycle crosses a critical surface.

6.3.3 The bubble equations and closed timeline curves in general

As we noted above, to avoid closed time-like curves in the neighborhood of the charge centres, one must require (6.35). A rather tedious computation shows that this condition may be rewritten as:

$$\sum_{\substack{j=1 \\ j \neq i}}^N \frac{\Gamma_{ij}}{|\vec{y}^{(j)} - \vec{y}^{(i)}|} = -2 \left(m_0 q_i + \frac{1}{2} \sum_{I=1}^3 k_i^I \right), \quad (6.37)$$

where

$$\Gamma_{ij} \equiv q_i q_j \Pi_{ij}^{(1)} \Pi_{ij}^{(2)} \Pi_{ij}^{(3)}. \quad (6.38)$$

The equations (6.37) are known as the “bubble equations” [10, 11, 13] or, in the four-dimensional context, “integrability conditions” [27, 35]. They impose an extra $(N-1)$ conditions¹¹ and relate the magnitudes of the magnetic fluxes to the geometric size of the configuration. The moduli space of these solutions is thus $2(N-1)$ -dimensional and consists of the relative positions of the $\vec{y}^{(i)}$ modulo the $(N-1)$ constraints (6.37).

Another potential source of CTC’s is the possibility of Dirac-Misner strings in ω . However, from (6.17) it is relatively easy to see that the absence of Dirac-Misner strings is equivalent to (6.35) or (6.37).

As noted in Section 3, the complete metric is stably causal if the t coordinate provides a global time function. In particular, the condition (3.8) and (3.10) now reduce to [11]:

$$-g^{\mu\nu} \partial_\mu t \partial_\nu t = -g^{tt} = (ZV)^{-1} (\mathcal{Q} - \omega^2) > 0, \quad (6.39)$$

where ω is squared using the \mathbb{R}^3 metric.

¹¹The sum of the bubble equations is a trivial identity given (6.22).

6.4 Mass, charge and angular momenta

As we noted earlier, the spatial metric will be asymptotically flat if the function, V , behaves asymptotically as $V \sim \frac{1}{r}$. The transformation to spherical polar coordinates at infinity involves setting $r = \frac{1}{4}\rho^2$ and one may then read off the ADM mass and the central charges from the coefficients of ρ^{-2} in g_{00} and the electrostatic potentials as described in Section 5.3.1.

The electrostatic potentials in (4.5) are simply, $-Z_I^{-1}$, and hence the asymptotic charges, Q_I , are read off from the expansion:

$$Z_I \sim 1 + \frac{Q_I}{4r} + \dots, \quad r \rightarrow \infty. \quad (6.40)$$

These functions have a rather nice expression in terms of the topological fluxes

$$Z_I V = V - \frac{1}{4} C_{IJK} \sum_{i,j=1}^N \Pi_{ij}^{(J)} \Pi_{ij}^{(K)} \frac{q_i q_j}{r_i r_j}, \quad (6.41)$$

from which it immediately follows that

$$Q_I = -C_{IJK} \sum_{i,j=1}^N q_i q_j \Pi_{ij}^{(J)} \Pi_{ij}^{(K)}. \quad (6.42)$$

This makes the role of the Chern-Simons terms evident in sourcing the electric charge: the electric charges are quadratics in the topological magnetic fluxes.

Expanding g_{00} , one has:

$$-g_{00} = (Z_1 Z_2 Z_3)^{-\frac{2}{3}} \sim 1 - \frac{2}{3} \sum_{I=1}^3 \frac{Q_I}{4r}, \quad (6.43)$$

and comparing this with (5.24) one finds

$$M = \frac{\pi}{4G_5} (Q_1 + Q_2 + Q_3). \quad (6.44)$$

It is fairly common to go to a system of units in which the five-dimensional Planck length, ℓ_5 , is unity and this means (see, for example, [98, 99]):

$$G_5 = \frac{\pi}{4}. \quad (6.45)$$

In particular, this means that the solution BPS condition takes the simpler standard form:

$$M = Q_1 + Q_2 + Q_3. \quad (6.46)$$

For completeness we also note that the angular momenta can be read off from the asymptotic expansion of the angular momentum vector, k :

$$k \sim \frac{1}{4\rho^2} ((J_1 + J_2) + (J_1 - J_2) \cos \theta) d\psi + \dots, \quad (6.47)$$

where θ is the polar angle in the flat \mathbb{R}^3 factor in (6.1). One then finds a combinatorial formula for $J_R \equiv J_1 + J_2$:

$$J_R \equiv J_1 + J_2 = \frac{4}{3} C_{IJK} \sum_{j=1}^N q_j^{-2} \tilde{k}_j^I \tilde{k}_j^J \tilde{k}_j^K, \quad (6.48)$$

where

$$\tilde{k}_j^I \equiv k_j^I - q_j N k_0^I, \quad \text{and} \quad k_0^I \equiv \frac{1}{N} \sum_{j=1}^N k_j^I. \quad (6.49)$$

Note that as a consequence of taking $q_0 = +1$ in (6.8) one has

$$\sum_{j=1}^N \tilde{k}_j^I = 0. \quad (6.50)$$

We also note that in terms of the parameters \tilde{k}_j^I , the expression for the charges takes on a more diagonal form:

$$Q_I = -2 C_{IJK} \sum_{j=1}^N q_j^{-1} \tilde{k}_j^J \tilde{k}_j^K. \quad (6.51)$$

The angular momentum, $J_L \equiv J_1 - J_2$, depends upon the details of the geometric configuration and may be thought of as a sum of dipole contributions coming from each 2-cycle:

$$\vec{J}_L \equiv J_1 - J_2 = \sum_{\substack{i,j=1 \\ j \neq i}}^N \vec{J}_{Lij}, \quad (6.52)$$

where

$$\vec{J}_{Lij} \equiv -\frac{4}{3} q_i q_j C_{IJK} \Pi_{ij}^{(I)} \Pi_{ij}^{(J)} \Pi_{ij}^{(K)} \hat{y}_{ij}, \quad (6.53)$$

and \hat{y}_{ij} are the *unit* vectors:

$$\hat{y}_{ij} \equiv \frac{(\vec{y}^{(i)} - \vec{y}^{(j)})}{|\vec{y}^{(i)} - \vec{y}^{(j)}|}. \quad (6.54)$$

If one thinks of

$$Q_{Iij} \equiv -\frac{1}{4} C_{IJK} q_i q_j \Pi_{ij}^{(J)} \Pi_{ij}^{(K)} \quad (6.55)$$

as the contribution to the charges (6.42) coming from individual bubbles, then [11, 33, 30]

$$\vec{J}_{Lij} = -\frac{16}{3} \sum_{I=1}^3 Q_{Iij} \Pi_{ij}^{(I)} \hat{y}_{ij}, \quad (6.56)$$

which shows that this part of the angular momentum comes from the $\mathbf{E} \times \mathbf{B}$ interactions on each of the bubbles.

The fact that the individual cycles may be viewed as carrying angular momentum lies at the heart of the semi-classical quantization of the moduli space of solutions [32, 36, 37].

6.5 Scaling solutions

While the special properties of scaling solutions is not our main focus here, it would be remiss of us not to give a brief review of the salient features of this class of solutions since they represent the physically most interesting microstate geometries and in particular they are the ones that closely approximate extremal black holes.

Scaling solutions arise whenever there is a set of points, \mathcal{S} , for which the bubble equations admit homogeneous solutions [27, 35, 30, 31, 32]:

$$\sum_{\substack{j \in \mathcal{S} \\ j \neq i}} \frac{\Gamma_{ij}}{|\vec{y}^{(j)} - \vec{y}^{(i)}|} = 0, \quad i \in \mathcal{S}. \quad (6.57)$$

It then follows that such a cluster of points can be scaled:

$$\vec{y}^{(j)} - \vec{y}^{(i)} \rightarrow \lambda (\vec{y}^{(j)} - \vec{y}^{(i)}), \quad (6.58)$$

for $\lambda \in \mathbb{R}$, and one can then examine the limit in which $\lambda \rightarrow 0$.

The geometries are, of course, required to satisfy (6.37) and not (6.57), however, given a solution of (6.57) one can easily make infinitesimal perturbations of the points, $\vec{y}^{(i)}$, and if $|\vec{y}^{(j)} - \vec{y}^{(i)}|$ is sufficiently small this will generate finite terms on the right-hand side of (6.57) and these can be used to generate solutions to the full bubble equations (6.37). In this way, the moduli space of physical solutions that satisfy (6.37) can contain scaling solutions in which a set of points, \mathcal{S} , can approach one another arbitrarily closely.

The simplest example of this kind of behaviour comes from scaling triangles. Suppose that $|\Gamma_{ij}|$, $i, j = 1, 2, 3$, satisfy the triangle inequalities:

$$|\Gamma_{13}| < |\Gamma_{12}| + |\Gamma_{23}| \quad \text{and cyclic}, \quad (6.59)$$

which means that we may arrange the points so that

$$|\vec{y}^{(j)} - \vec{y}^{(i)}| = \lambda |\Gamma_{ij}|, \quad (6.60)$$

for $\lambda \in \mathbb{R}^+$. The fluxes can then usually be arranged so that the homogeneous bubble equations, (6.57), are trivially satisfied since they amount to $\pm\lambda^{-1} \mp \lambda^{-1} = 0$. When the triangle has infinitesimal size, making infinitesimal deformations of the angles can be used to generate solutions to the original bubble equations (6.37). In particular, in a physical solution to (6.37) with three fluxes that obey (6.59), one can make the three points approach one another arbitrarily closely by adjusting the angles in the triangle so that they approach the angles in the triangle defined by (6.60).

The existence of scaling solutions to the bubble equations, or integrability conditions, was first noted in [27, 35]. However, this seemed to be a rather singular limit but it was subsequently

shown that, from the perspective of five-dimensional supergravity, this limit is not only non-singular but also defines perhaps the most important class of physical solutions [30, 31, 32].

Suppose that we have a scaling cluster, \mathcal{S} , that is centred on the origin, $r = 0$. Let ϵ be the largest separation (in \mathbb{R}^3) between points in \mathcal{S} and let η be the smallest distance from a point in \mathcal{S} and a point, $\vec{y}^{(i)}$, that not in \mathcal{S} . Assume that $\epsilon \ll \eta$ and, for simplicity, suppose that the total geometric charge of the cluster is unity: $q_{\mathcal{S}} \equiv \sum_{i \in \mathcal{S}} q_i = 1$. In the intermediate range of r in which, $\epsilon \ll r \ll \eta$, one has $V \sim \frac{1}{r}$ and all the other functions K_I and L_I behave as $\mathcal{O}(r^{-1})$. This means that, in the intermediate region, $Z_I \sim \frac{Q_{I,\mathcal{S}}}{4r}$. where the $Q_{I,\mathcal{S}}$ are the electric charges associated with the scaling cluster. Using this in (4.3) and (6.1) we see that the metric in the intermediate region becomes:

$$ds_5^2 = -\frac{16r^2}{a^4}(dt+k)^2 + \frac{a^2}{4}\frac{dr^2}{r^2} + \frac{a^2}{4}[(d\psi + \cos\theta d\phi)^2 + d\theta^2 + \sin^2\theta d\phi^2], \quad (6.61)$$

where $a = (Q_{1,\mathcal{S}}Q_{2,\mathcal{S}}Q_{3,\mathcal{S}})^{1/6}$. This is the metric of an $AdS_2 \times S^3$ throat of a rotating, extremal black hole.

There are several important consequences of this result. First, such scaling clusters look almost exactly like extremal black holes except that they “cap off” in a collection of bubbles just above¹² where the horizon would be for the extremal black hole. Moreover, while it appears, from the perspective of the \mathbb{R}^3 base, that the bubbles are collapsing in the scaling limit, they are, in fact, simply creating an AdS throat and descending down it as it forms. The physical size of the bubbles approaches a large, finite value whose scale is set by the radius, a , of the S^3 of the throat, which corresponds to the horizon of the would-be black hole. Thus the scaling microstate geometries represent deep bound states of bubbles that realize the goal of creating a smooth, solitonic solutions that look like BPS black holes. One obtains similar results for black rings from scaling clusters whose net geometric charge, $q_{\mathcal{S}}$, is zero.

The fact that one can adjust classical parameters so that the scaling points approach one another arbitrarily closely means that the AdS throat can be made arbitrarily deep. However, the angular momentum (6.52) depends, via (6.53), upon the details of locations of the points and when angular momentum is quantized this will lead to a discretization of the moduli space and will limit the depth of simple scaling solutions like those based on scaling triangles [32]. More generally, it was proposed in [32] and then proven in [36] that the individual contributions, \vec{J}_{Lij} in (6.53) must be separately quantized and so, upon quantization, the classical moduli space is completely discrete. This has the very interesting physical consequence that even though very long, deep throats are macroscopic regions of space time in which the curvature length scale can be uniformly bounded to well above the Planck scale, quantum effects can wipe out such regions of space-time.

¹²From the perspective of an infalling observer.

6.6 Some simple examples

6.6.1 Two centres: $AdS_3 \times S^2$

Our first example [100, 101, 102] has equal and opposite geometric charges and $\varepsilon_0 = 0$ in (6.18). We now show that the metric is that of $AdS_3 \times S^2$. More generally, if the total geometric charge, q_0 in (6.8), vanishes then the metric will be asymptotic to global $AdS_3 \times S^2$. In particular, we make contact with the discussion at the end of Section 3.

We locate the two centres on the z -axis at $z = \pm a$ and define:

$$r_{\pm} \equiv \sqrt{\rho^2 + (z \mp a)^2}, \quad (6.62)$$

where (z, ρ, ϕ) are cylindrical polar coordinates on the \mathbb{R}^3 base. Take the harmonic functions to be

$$V = \left(\frac{1}{r_+} - \frac{1}{r_-} \right), \quad K = k \left(\frac{1}{r_+} + \frac{1}{r_-} \right), \quad (6.63)$$

$$L = -k^2 \left(\frac{1}{r_+} - \frac{1}{r_-} \right), \quad M = -\frac{2k^3}{a} + \frac{1}{2}k^3 \left(\frac{1}{r_+} + \frac{1}{r_-} \right), \quad (6.64)$$

where the constant in M has been chosen so as to make the metric regular at infinity.

The vector potentials for this solution are then:

$$A = \left(\frac{(z-a)}{r_+} - \frac{(z+a)}{r_-} \right) d\phi, \quad \omega = -\frac{2k^3}{a} \frac{\rho^2 + (z-a+r_+)(z+a-r_-)}{r_+ r_-} d\phi. \quad (6.65)$$

The five-dimensional metric is then:

$$ds_5^2 \equiv -Z^{-2} (dt + \mu(d\psi + A) + \omega)^2 + Z (V^{-1}(d\psi + A)^2 + V(d\rho^2 + \rho^2 d\phi^2 + dz^2)), \quad (6.66)$$

where

$$Z = V^{-1}K^2 + L = -\frac{4k^2}{(r_+ - r_-)}, \quad (6.67)$$

$$\mu = V^{-2}K^3 + \frac{3}{2}V^{-1}KL + M = 4k^3 \frac{(r_+ + r_-)}{(r_+ - r_-)^2} - \frac{2k^3}{a}.$$

To map this onto a more standard form of $AdS_3 \times S^2$ one must make a transformation to oblate spheroidal coordinates like those employed in [103] to map positive-definite two-centred space onto the Eguchi-Hanson form:

$$z = a \cosh 2\xi \cos \theta, \quad \rho = a \sinh 2\xi \sin \theta, \quad \xi \geq 0, \quad 0 \leq \theta \leq \pi. \quad (6.68)$$

In particular, one has $r_{\pm} = a(\cosh 2\xi \mp \cos \theta)$. One then rescales and shifts the remaining variables according to:

$$\tau \equiv \frac{a}{8k^3} t, \quad \varphi_1 \equiv \frac{1}{2} \psi - \frac{a}{8k^3} t, \quad \varphi_2 \equiv \phi - \frac{1}{2} \psi + \frac{a}{4k^3} t, \quad (6.69)$$

and the five-dimensional metric takes the standard $AdS_3 \times S^2$ form:

$$ds_5^2 \equiv R_1^2 [-\cosh^2 \xi d\tau^2 + d\xi^2 + \sinh^2 \xi d\varphi_1^2] + R_2^2 [d\theta^2 + \sin^2 \theta d\varphi_2^2], \quad (6.70)$$

with

$$R_1 = 2R_2 = 4k. \quad (6.71)$$

Note that the first factor in the metric is *global* AdS_3 with $-\infty < \tau < \infty$.

Note that

$$g_{\psi\psi} = \frac{1}{4} R_1^2 \sinh^2 \xi + \frac{1}{4} R_2^2 \sin^2 \theta. \quad (6.72)$$

Thus the Killing field, $\frac{\partial}{\partial \psi}$ has two fixed points, or nuts, at $\xi = 0, \theta = 0$ and $\xi = 0, \theta = \pi$, that is, at $r_{\pm} = 0$. It follows that the base manifold, \mathcal{B} , has Euler characteristic 2, consistent with its topology being $\mathbb{R}^2 \times S^2$.

Also note that the time-like Killing vector, $\mathbf{K} = \frac{\partial}{\partial t}$, of the original space-time metric (4.3) is related, in this example, to the Killing vectors of $AdS_3 \times S^2$ by (3.11). In this sense, the only unusual feature of the critical surface, $V = 0$, is the behaviour of the family of “observers” with worldlines defined by the integral curves of $\mathbf{K} = \frac{\partial}{\partial t}$. More generally, critical surfaces occur between pairs of geometric charges of opposite sign and the smoothness of the solution across critical surfaces will follow for the same basic reason that we have described here.

6.6.2 Three centres

We now consider an example in which $\varepsilon_0 = 0$ again but the total geometric charge, q_0 in (6.8), is 1. This means that the space-time is asymptotically Euclidean.

Take

$$V = \frac{1}{r_+} - \frac{1}{r_0} + \frac{1}{r_-}, \quad (6.73)$$

and

$$K^I = K = \frac{k}{r_+} + \frac{k}{r_0} + \frac{k}{r_-}, \quad (6.74)$$

where

$$r_{\pm} = \sqrt{\rho^2 + (z \mp 1)^2}, \quad r_{\pm} = \sqrt{\rho^2 + z^2}. \quad (6.75)$$

The remaining functions are then

$$L_I = L = 1 - \frac{k^2}{r_+} + \frac{k^2}{r_0} - \frac{k^2}{r_-} = 1 - k^2 V, \quad (6.76)$$

$$M = -\frac{9}{2}k + \frac{1}{2} \left(\frac{k^3}{r_+} + \frac{k^3}{r_0} + \frac{k^3}{r_-} \right) = -\frac{9}{2}k + \frac{1}{2}k^2 K, \quad (6.77)$$

Normally (6.37) gives complicated rational equations relating flux parameters to positions but for this problem it collapses to:

$$k = \frac{\sqrt{3}}{2}. \quad (6.78)$$

One then has

$$ZV = K^2 + LV = V + \frac{4k^2}{r_0} \left(\frac{1}{r_-} + \frac{1}{r_+} \right) \quad (6.79)$$

$$= \frac{1}{r_+} - \frac{1}{r_0} + \frac{1}{r_-} + \frac{3}{r_0} \left(\frac{1}{r_-} + \frac{1}{r_+} \right) \quad (6.80)$$

$$= \frac{r_0 - r_+ + 1}{r_0 r_+} + \frac{1}{r_-} + \frac{3}{r_- r_0} + \frac{2}{r_0 r_+} \quad (6.81)$$

The first term is positive definite because of the triangle inequality and so one has $ZV > 0$ globally.

Set

$$V_0 \equiv \frac{1}{r_0}, \quad V_1 \equiv \frac{1}{r_+} + \frac{1}{r_-}, \quad (6.82)$$

then μ in (6.15) can be simplified to

$$\mu = V^{-2} (4k^3 V_0 V_1 (V_0 + V_1) + 3k (V_1 - V_0) (2V_0 - V_1)), \quad (6.83)$$

and \mathcal{Q} may be written

$$\mathcal{Q} = -\frac{27}{4} (V_0 V_1 - V_1 - 2V_0)^2 + 63 V_0 V_1 + (V_1 - V_0). \quad (6.84)$$

Finally, ω in (6.15) can be written:

$$\omega = -\frac{4k^3}{r_0} \left[\frac{1}{r_+} (\rho^2 + (z - r_0)(z - 1 + r_+)) - \frac{1}{r_-} (\rho^2 + (z + r_0)(z + 1 - r_-)) \right] d\phi. \quad (6.85)$$

Observe that ω vanishes identically on the z -axis ($\rho = 0$) as it must if one is to avoid CTC's (indeed it vanishes quadratically in ρ).

If one plots $\frac{\mathcal{Q}}{Z^2 V^2}$, as shown in Fig. 1, one easily see that it is non-negative and vanishes only when $\rho = 0$ and either $z = 0$ or $z = \pm 1$. Thus (6.84) has the proper behaviour.

If one plots $(ZV)^{-1}(\mathcal{Q} - \omega^2)$, as shown in Fig. 2, one may verify that it is strictly positive and regular and so (6.39) is satisfied. One finds that g^{tt} has discontinuous first derivatives in z at $z = 0$ and $z = \pm 1$ for $\rho = 0$. These are coordinate artifacts inherited from the fact that ρ and z are not good coordinates in the neighborhood of $\rho = 0$ and $z = 0, \pm 1$. These apparent kinks are removed by the usual $r_j \rightarrow \frac{1}{4} R_j^2$ change of variable.

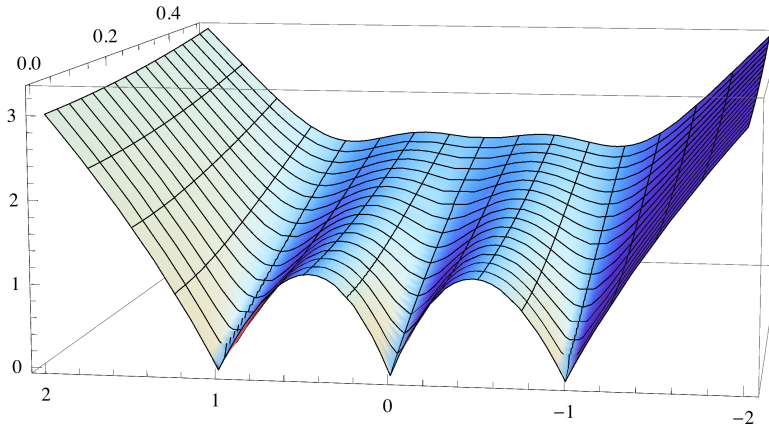


Figure 1: Graph of $\frac{Q}{z^2 V^2}$ for $-2 \leq z \leq 2$, $0 \leq \rho \leq 0.5$. Observe that it is globally non-negative.

7 A non-BPS Soliton that violates the BPS bound

There are only rather few known examples of non-BPS, non-extremal solitons [104, 105, 106, 107, 108, 109] and most of these involve a single topological bubble. The difficulty is, of course, largely because one has to solve equations of motion rather than the much simpler BPS equations. Here we will re-analyse a hitherto puzzling “soliton” solution of five-dimensional minimal supergravity [110] using the methods and ideas of the present paper. The approach to finding this soliton is rather reminiscent of the construction of the soliton in [104, 105, 106] except that the solution of [104] is only regular in six dimensions while the solution of [105, 106, 110] are smooth in five-dimensions. A striking feature of the solutions of [110] is that they violate the BPS bound for the mass in terms of the electric charge (a proof of which for minimal supergravity using spinors was given in [95]). We shall argue that this anomalous behaviour arises because this solitonic spacetime does not admit a spin structure.

7.1 The solution

The Lagrangian of five-dimensional minimal supergravity is obtained from (4.1) by setting all the vector fields equal¹³, $A^I = \frac{1}{\sqrt{3}}A$, and trivializing the scalars by setting $X^I = 1$.

The starting point for constructing the soliton is the black hole solution of [111] in gauged minimal supergravity, which we will immediately specialize to the ungauged theory by setting the gauge coupling, g , to zero. If $g = 0$ then it is claimed in [110] that regularity requires that one set the parameters a and b of [111] to be of equal magnitude and the parameter q must be

¹³With these choices of normalization our five-dimensional action matches precisely the ungauged action in [110].

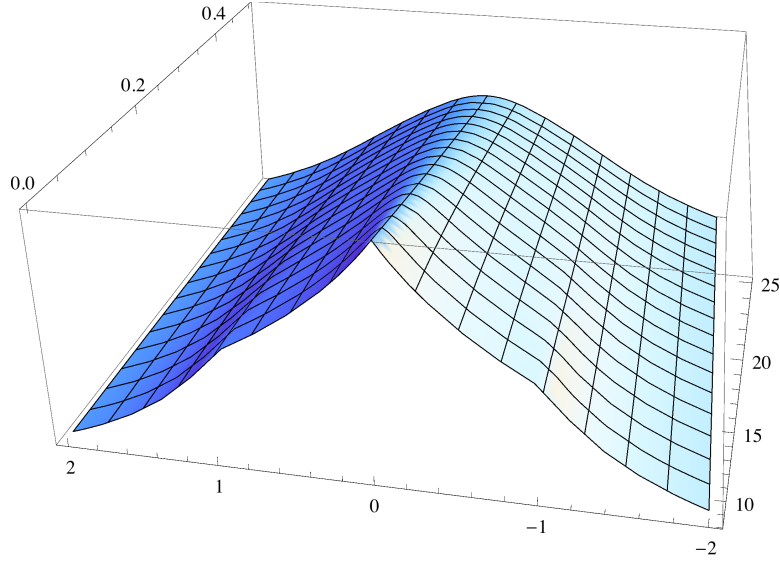


Figure 2: Graph of $-g^{tt}$ given in (6.39) for $-2 \leq z \leq 2$, $0 \leq \rho \leq 0.5$. Observe that $-g^{tt} > 0$ everywhere and so the space-time is stably causal. The cusps at $\rho = 0$, $z = \pm 1$ and $\rho = 0$, $z = 0$ are a coordinate artifacts arising from the fact that ρ and z are not good coordinates at these points.

negative. We choose $a = b$ and the metric then becomes:

$$ds^2 = - \left(1 - \frac{2m}{\rho^2} + \frac{q^2}{\rho^4}\right) - \frac{2}{\rho^2} \left((q + 2m) - \frac{q^2}{\rho^2}\right) dt \omega + \frac{1}{\rho^2} \left(2(q + m) - \frac{q^2}{\rho^2}\right) \omega^2 + \frac{\rho^2 r^2 dr^2}{W} + \rho^2 (d\theta^2 + \sin^2 \theta d\phi_1^2 + \cos^2 \theta d\phi_2^2), \quad (7.1)$$

where

$$\rho^2 \equiv r^2 + a^2, \quad \omega \equiv a(\sin^2 \theta d\phi_1 + \cos^2 \theta d\phi_2), \quad W \equiv \rho^4 + q^2 + 2a^2 q - 2mr^2, \quad (7.2)$$

The soliton is found by continuing to negative values of r^2 and so we define $X = \rho^2 = r^2 + a^2$ so that $2rdr = dX$. The spatial metric (obtained by setting $t = 0$) is, in fact, of cohomogeneity one and admits an action of $U(2)$ whose principal orbits $X = \text{constant}$ are squashed 3-spheres. To make this manifest we define $\phi_1 = (\psi + \phi)/2$ and $\phi_2 = (\psi - \phi)/2$ and introduce the standard left-invariant one-forms noting that $\omega = \frac{a}{2}\sigma_3$. The metric is therefore given by:

$$ds^2 = \frac{XdX^2}{4(X^2 - 2mX + 2a^2(q + m) + q^2)} + \frac{X}{4}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - \left(1 - \frac{2m}{X} + \frac{q^2}{X^2}\right) dt^2 - 2\left(\frac{2m + q}{X} - \frac{q^2}{X^2}\right) a \sigma_3 dt + \frac{a^2}{4}\left(2\frac{m + q}{X} - \frac{q^2}{X^2}\right) \sigma_3^2. \quad (7.3)$$

The Maxwell field is relatively simple:

$$A = q \frac{\sqrt{3}}{X} \left(dt - \frac{a}{2} \sigma_3\right). \quad (7.4)$$

We define

$$Q \equiv -\frac{q}{a^2} \quad (7.5)$$

and note that, according to [110], regularity requires that $Q > 0$. In [110] it is also pointed out that if m is chosen so that

$$m = -\frac{1}{2}q(1 + \sqrt{Q}) = \frac{1}{2}a^2Q(1 + \sqrt{Q}), \quad (7.6)$$

with

$$0 < -a^2q < a^4 \quad \Leftrightarrow \quad 0 < Q < 1, \quad (7.7)$$

then one can arrange that

$$g_{\psi t}(X_0) = 0, \quad g^{XX}(X_0) = 0, \quad g_{\psi\psi}(X_0) = 0 \quad (7.8)$$

at

$$X = X_0 \equiv \sqrt{-a^2q} = a^2\sqrt{Q}. \quad (7.9)$$

Note that for this to happen, q must be negative and that m and $2m + q$ are both positive but $m + q$ is negative. The position, X_0 , is also positive but at $X = X_0$, r^2 is negative.

In the obvious notation we indeed find the following

$$g_{tt} = -\left(1 - \frac{2m}{X} + \frac{q^2}{X^2}\right) = -X^{-2}(X^2 - a^2Q(1 + \sqrt{Q})X + a^4Q^2), \quad (7.10)$$

$$g_{\psi t} = a - \left(\frac{2m + q}{X} - \frac{q^2}{X^2}\right) = -X^{-2}(a\sqrt{Q})^3(X - a^2\sqrt{Q}), \quad (7.11)$$

$$g_{\psi\psi} = \frac{X}{4} + \frac{a^2}{4}\left(2\frac{m + q}{X} - \frac{q^2}{X^2}\right) = -X^{-2}(X - a^2\sqrt{Q})(X^2 + a^2\sqrt{Q}X + a^4Q\sqrt{Q}), \quad (7.12)$$

$$g^{XX} = 4X^{-1}(X^2 - 2mX + 2a^2(q + m) + q^2) \quad (7.13)$$

$$= 4X^{-1}(X - a^2\sqrt{Q})(X + a^2\sqrt{Q}(1 - \sqrt{Q} - Q)) \quad (7.14)$$

and

$$g^{tt} = \frac{g_{\psi\psi}}{g_{tt}g_{\psi\psi} - g_{\psi t}^2} = \frac{(X^2 + a^2\sqrt{Q}X + a^4Q\sqrt{Q})}{X(X + a^2\sqrt{Q}(1 - \sqrt{Q} - Q))}. \quad (7.15)$$

The fact that $g_{\psi t}(X_0) = g_{\psi\psi}(X_0) = 0$ means that the ψ -fibre is pinching off at $X = X_0$ leaving a finite sized S^2 “bolt,” or “bubble.” Smoothness requires that we make sure that the metric has no conical singularities at the bolt and so we need to examine the metric in the neighborhood of $X = X_0$.

7.2 Regularity and properties of the solution

7.2.1 Removal of conical singularities

Setting $X = X_0 + x$ and expanding x , the spatial metric transverse to the bolt has the form

$$ds_2^2 = u \frac{dx^2}{x} + v x d\psi^2, \quad (7.16)$$

where

$$u = g'_{XX}(X_0) = \frac{1}{8} \left(1 - \frac{m}{X_0}\right)^{-1} = \frac{1}{4(2 - \sqrt{Q} - Q)}, \quad (7.17)$$

$$v = g'_{\psi\psi}(X_0) = \frac{1}{4} \left(1 - 2a^2 \frac{m+q}{X_0^2} + 2a^2 \frac{q^2}{X_0^3}\right) = \frac{1}{4}(2 + \sqrt{Q}), \quad (7.18)$$

and $'$ denotes differentiation with respect to X .

If we define a radial variable

$$y = 2\sqrt{ux} \quad (7.19)$$

we find

$$ds_2^2 = dy^2 + y^2 \frac{v}{4u} d\psi^2 \quad (7.20)$$

and therefore in order to eliminate a conical singularity we require

$$\psi \in (0, 4\pi \sqrt{\frac{u}{v}}] \quad (7.21)$$

On the other hand asymptotic flatness requires

$$\psi \in (0, 4\pi] \quad (7.22)$$

and thus consistency requires

$$\frac{v}{u} = (2 + \sqrt{Q})(2 - \sqrt{Q} - Q) = 1. \quad (7.23)$$

This cubic has a unique solution in the range $0 < Q < 1$ and it is at $Q \approx 0.7733184\dots$.

7.2.2 Ergo-regions and stable causality

From (7.10) one can write

$$-g_{tt} = X^{-2}((X - a^2 Q)^2 + Q(1 - \sqrt{Q})X) \quad (7.24)$$

which is manifestly positive for $0 < Q < 1$ and hence there are no ergo-regions.

Moreover, from (7.15) one can easily see that $g^{tt} > 0$ for $X > X_0 = a^2 \sqrt{Q}$, with $0 < Q < 1$ and hence the space-time is stably causal.

7.2.3 The Field strength

The field strength $F = dA$ with A given by (7.4) is simply

$$F = -q\sqrt{3}X^{-2}dX \wedge \left(dt - \frac{a}{2}\sigma_3\right) - 2\sqrt{3}aqX^{-1}\sigma_1 \wedge \sigma_2. \quad (7.25)$$

Since the 2-sphere always has finite size for $X \geq X_0$, the second term is globally regular and gives a finite contribution to

$$F^2 \equiv F_{\mu\nu}F^{\mu\nu}. \quad (7.26)$$

The first term in (7.25) is similarly smooth, but it might give a divergent contribution to F^2 because the fibre is pinching off. One can easily check that $g^{\psi\psi}$ diverges as $(X - X_0)^{-1}$ at $X = X_0$, however from (7.14) one sees that g^{XX} vanishes as $(X - X_0)$ and so the contribution to F^2 is also finite. Thus F is well behaved.

7.2.4 Mass and Charge: Violation of the BPS bound

According to [110] the mass M and electric charge Q_E are given by

$$M = \frac{3\pi m}{4G}, \quad Q_E = -\frac{\sqrt{3}\pi a^2 Q}{4G} \quad (7.27)$$

Thus

$$\frac{\sqrt{3}Q_E}{M} = -\frac{2}{1 + \sqrt{Q}} \quad (7.28)$$

which is in contradiction with the usual BPS bound [95]

$$M \geq \sqrt{3}|Q_E|. \quad (7.29)$$

This seems very mysterious but the explanation lies in the topology and the fact that the space-time does not have a spin structure, as we will now demonstrate.

Since the space-time is stably causal, the local section, or base \mathcal{B} , given by $t = \text{constant}$ is a Cauchy surface on which the induced metric:

$$ds_4^2 = \frac{XdX^2}{4(X^2 - 2mX + 2a^2(q + m) + q^2)} + \frac{X}{4}(\sigma_1^2 + \sigma_2^2) + \frac{1}{4}\left(X + a^2\left(2\frac{m+q}{X} - \frac{q^2}{X^2}\right)\right)\sigma_3^2 \quad (7.30)$$

is positive definite.

Since there is an action of $U(2)$ with a two-dimensional fixed point set, we can compare with the $U(2)$ invariant form of the Fubini-Study metric on \mathbb{CP}^2 given in [112]

$$ds^2 = \frac{dr^2}{1 + \frac{\Lambda r^2}{6}} + \frac{r^2}{4(1 + \frac{\Lambda r^2}{6})^2}\sigma_3^2 + \frac{r^2}{4(1 + \frac{\Lambda r^2}{6})}(\sigma_1^2 + \sigma_2^2), \quad (7.31)$$

where Λ is the cosmological constant. If we set $r = \frac{1}{X-X_0}$ then the origin, $r = 0$, of \mathbb{CP}^2 maps to infinity $X = \infty$ and the bolt, or complex line, at $r = \infty$ maps to the bolt at $X = X_0$. Thus the base, \mathcal{B} , has the topology, but of course not the metric, of $\mathbb{CP}^2 \setminus 0$. The bolt, or bubble, corresponds to a complex line \mathbb{CP}^1 in \mathbb{CP}^2 . Notoriously every line in the complex projective plane intersects every other line once and only once and therefore the self-intersection number of the bolt is unity, which is an odd number, and so the second Stiefel-Whitney class of \mathcal{B} is non-vanishing and hence the spacetime does not admit a spin structure.

This presumably accounts for the breakdown of the spinorial proof given in [95] of the BPS bound.

8 Topology, Fluxes, Charge and the Smarr Formula for BPS Solitons

8.1 The Smarr formula revisited

We now apply the results of Section 5 to the general class of examples that we have just described. First, it follows immediately from (4.5) and (5.13) that

$$\lambda^I = Z_I^{-1} - \alpha^I, \quad (8.1)$$

where α^I are constants. Indeed, with the boundary conditions (4.11), one should take the $\alpha^I = 1$. It is only with this choice that the boundary terms at infinity involving λ^I can be dropped in (5.37) to arrive at the result (5.38) and ultimately at (5.40). In spite of this, we will retain α^I because they will prove useful in elucidating the computation.

One can then show that

$$i_K G_I + \frac{1}{2} C_{IJK} \lambda^J F^K = -\frac{1}{2} d(Z_I Z^{-3} (dt + k)) - \frac{1}{2} C_{IJK} \alpha^J F^K, \quad (8.2)$$

where one should recall the definition of Z in (4.7). Hence one has

$$\Lambda_I = -\frac{1}{2} d(Z_I Z^{-3} (dt + k)), \quad H_I = -\frac{1}{2} C_{IJK} \alpha^J F^K. \quad (8.3)$$

We must show that the vector field, Λ_I , is globally well-defined and smooth. There are several potential sources of singular behaviour: (i) if one of the Z_J vanishes, (ii) when $V = 0$ and (iii) when $r_j = 0$. First we note that $Z_I V > 0$ and Z_I is finite when $r_j = 0$ and thus Z_I never vanishes. Since Z_I is finite when $r_j = 0$, and the bubble equations require that k vanishes when $r_j = 0$, there are no singularities at $r_j = 0$. Finally, near $V = 0$, the vector k diverges as V^{-2} but $Z_I Z^{-3} \sim \mathcal{O}(V^2)$ and so Λ_I is smooth in a neighborhood of these surfaces.

The mass formula thus reduces to:

$$\begin{aligned} M &= \frac{1}{32\pi G_5} \int_{\Sigma} \left[Q^{IJ} H_{I\rho\sigma} G_J^{\rho\sigma\nu} \right] d\Sigma_{\nu} = -\frac{1}{64\pi G_5} C_{ILM} \alpha^L \int_{\Sigma} \left[Q^{IJ} F_{\rho\sigma}^M G_J^{\rho\sigma\nu} \right] d\Sigma_{\nu} \\ &= -\frac{1}{128\pi G_5} C_{IJK} \alpha^I \int_{\Sigma} \epsilon_{\mu\alpha\beta\rho\sigma} F^{J\alpha\beta} F^{K\rho\sigma} d\Sigma^{\mu} \end{aligned} \quad (8.4)$$

$$= -\frac{1}{8\pi G_5} \int_{\Sigma} \nabla_{\rho} \left[\alpha^I Q_{IJ} F^{J\rho}_{\mu} \right] d\Sigma^{\mu} = -\frac{1}{8\pi G_5} \int_{\infty} \alpha^I Q_{IJ} F^J_{\mu\nu} d\Sigma^{\mu\nu}. \quad (8.5)$$

where, in the last line, we have used the Maxwell equations, (5.3). Note that this last expression precisely matches the term involving λ^I in (5.37) but with $\lambda^I \rightarrow \alpha^I$. Indeed, we may rewrite (5.37) as

$$\begin{aligned} M &= -\frac{1}{16\pi G_5} \int_{\Sigma} \nabla_{\rho} \left[2 Q_{IJ} (\lambda^I + \alpha^I) F^{J\rho\mu} + Q^{IJ} \Lambda_{I\sigma} G_J^{\sigma\rho\mu} \right] d\Sigma_{\mu} \\ &= -\frac{1}{16\pi G_5} \int_{\Sigma} \nabla_{\rho} \left[2 Q_{IJ} (Z_I^{-1}) F^{J\rho\mu} + Q^{IJ} \Lambda_{I\sigma} G_J^{\sigma\rho\mu} \right] d\Sigma_{\mu}. \end{aligned} \quad (8.6)$$

where we have used (8.1).

We therefore see that the Smarr formula may be arranged, by taking $\alpha^I = 1$, so that the only contribution to the mass is the topological term (5.40) and hence (8.4). On the other hand, one could have chosen $\alpha^I = 0$, which kills the explicit topological term but leaves a boundary term at infinity in the original computation. This boundary term is explicitly exhibited in (8.5). Equation (8.6) gives the exact Smarr formula, independent of the choice of constant in (8.1).

Taking $\alpha^I = 1$ in (8.5) and noting that $Q_{IJ} \rightarrow \frac{1}{2} \delta_{IJ}$ at infinity gives:

$$M = -\frac{1}{16\pi G_5} \sum_{I=1}^3 \int_{\infty} F^I_{\mu\nu} d\Sigma^{\mu\nu} = \frac{1}{8\pi G_5} (2\pi^2) (Q_1 + Q_2 + Q_3) = \frac{\pi}{4G_5} (Q_1 + Q_2 + Q_3), \quad (8.7)$$

where the $2\pi^2$ comes from the volume of the unit S^3 . This result, of course, matches (6.44). The charges, Q_I , are defined by (6.40) and given by (6.42). Given the latter form the charges, it is evident that we could easily obtain the same result from the topological integral. More specifically, one can rewrite (8.4) as

$$M = -\frac{1}{32\pi G_5} C_{IJK} \alpha^I \int_{\Sigma} F^J \wedge F^K, \quad (8.8)$$

and in Section 8.3 we will evaluate this by using the intersection form.

8.2 The topology of the base

In the light of our discussion in Section 6.3, we describe the topology of the base manifold \mathcal{B} in greater detail. For more detail, see [113, 114, 115, 116]. This is determined by the following data, ε_0 , $q_0 = \sum_j q_j$ and $N = \sum_A |q_A|$. There are essentially three distinct possibilities

- If $\varepsilon_0 = 1$ the base manifold $\{\mathcal{B}, g_{\mu\nu}\}$ will be asymptotically locally flat, ALF, that is near infinity the manifold is, in general, a twisted circle bundle over $\mathbb{R} \times S^2$. If $q_0 = 0$, the bundle is untwisted and the base manifold $\{B, g_{\mu\nu}\}$ will be asymptotically flat, AF.
- If $\varepsilon_0 = 0$ the base manifold $\{\mathcal{B}, g_{\mu\nu}\}$ will, in general, be asymptotically locally Euclidean, ALE. That is, near infinity the manifold will approach the quotient $\mathbb{E}^4/C_{|q_0|}$, of Euclidean space by the cyclic group of order $|q_0|$. The action of the cyclic group is given by its embedding into one of the $SU(2)$ subgroups of $SO(4) \equiv (SU(2) \times SU(2))/\mathbb{Z}_2$. If $|q_0| = 1$, then base manifold $\{\mathcal{B}, g_{\mu\nu}\}$ will be asymptotically Euclidean, AE.
- If $\varepsilon_0 = 0$ and $q_0 = 0$, then the metric is asymptotic to global $AdS_3 \times S^2$.

The Euler number, $\chi(\mathcal{B})$, is simply the number of 2-cycles plus $b_0 = +1$ and so:

$$\chi(\mathcal{B}) = N = \sum_A |q_A|. \quad (8.9)$$

A basis c_A , $A = 1, \dots, N-1$ for the second homology, $H_2(\mathcal{B}, \mathbb{Z})$, given by an ordering of the points such that the vectors $\vec{y}^{(A+1)} - \vec{y}^{(A)}$ and $\vec{y}^{(B+1)} - \vec{y}^{(B)}$ intersect only if $|A - B| = 1$, and then at only one point. The elements of the intersection form, I_{AB} , in this basis, are given by the cup product

$$I_{AB} = c_A \cup c_B. \quad (8.10)$$

One has that

$$\begin{aligned} I_{AB} &= \pm |A - B|, & \text{if } |A - B| = 1 \\ I_{AB} &= 0, & \text{if } |A - B| > 1 \\ I_{AA} &= q_A + q_{A+1}. \end{aligned} \quad (8.11)$$

The signs of the off-diagonal terms, I_{AB} , depend upon the relative orientation of the cycles A and B .

Then $\tau(\mathcal{B})$ is the signature and $\chi(\mathcal{B})$ the rank plus one of this matrix. Under a change of basis we have

$$I \rightarrow S^t I S, \quad S \in SL(N-1, \mathbb{Z}). \quad (8.12)$$

For an axisymmetric configuration, the points lie on a line and this provides a natural ordering for the cycles. Thus if $q_j = (+1, +1, -1)$ we have

$$I = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \quad (8.13)$$

while if $q_j = (+1, -1 + 1)$, we have

$$I = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (8.14)$$

but these are equivalent in $SL(2, \mathbb{Z})$ since

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (8.15)$$

The topology of \mathcal{B} is thus $S^2 \times S^2 \setminus \infty$, where ∞ is the point at infinity which is removed.

More generally, for $N = 2p + 1$ points with $|q_j| = 1$ and $q_0 = 1$, a basis may be found such that the intersection form, I_{AB} , is N direct summands of matrices of the form (8.14). Thus the rank is $2p$ and the signature is zero. Assuming that manifold to be simply connected, the topology must be the connected sum of p copies of $S^2 \times S^2$ with a point removed. For $N = 2p$ points with $|q_j| = 1$ and $q_0 = 0$, again assuming simple connectivity, the topology is $(R^2 \times S^2) \# (S^2 \times S^2) \# \dots \# (S^2 \times S^2)$. Thus every time you add a point and an anti-point you blow up an $S^2 \times S^2$.

8.3 Fluxes and charges

As noted earlier, representatives of the homology cycles are given by the ψ -fibration over any simple curve between two points, $\bar{y}^{(i)}$ and $\bar{y}^{(j)}$. We will thus denote homology cycles by pairs of points. To introduce a basis we will choose the labeling of the points, $\bar{y}^{(i)}$, so that

$$q_j = (-1)^{j+1} \quad (8.16)$$

and define a “simple root” basis for the homology cycles, c_A , by

$$c_A = \alpha_{A,j} \bar{y}^{(j)}, \quad \alpha_{A,j} = \delta_A^j - \delta_{A+1}^j, \quad A = 1, \dots, N-1. \quad (8.17)$$

Similarly, the cohomological fluxes can be parametrized by the vectors, k_j^I , appearing in the harmonic functions, K^I , in (6.18). Indeed, it is convenient to introduce a metric, \tilde{I}_{ij} , and its inverse, \tilde{I}^{ij} , defined by

$$\tilde{I}_{ij} = \text{diag}(q_1, q_2, \dots, q_N), \quad \tilde{I}^{ij} = \text{diag}(q_1^{-1}, q_2^{-1}, \dots, q_N^{-1}), \quad (8.18)$$

and then the basis fluxes of (6.30) are given by:

$$\Pi_{A+1 A}^{(I)} = \tilde{I}^{ij} \alpha_{A,i} k_j^I. \quad (8.19)$$

Define

$$I_{AB} = \tilde{I}^{ij} \alpha_{A,i} \alpha_{B,j}, \quad (8.20)$$

and by direct computation one can verify that

$$I_{AB} = I_{BA} = \begin{cases} -q_B^{-1} & \text{if } B = A + 1 \\ -q_A^{-1} & \text{if } A = B + 1 \\ 0 & \text{otherwise.} \end{cases} \quad (8.21)$$

Comparing with (8.11), we see that I_{AB} is the intersection matrix for our choice of homology basis. Indeed one can make $I_{AA+1} = I_{A+1A} = +1$ by reversing the orientations of every second cycle.

Let ν^A be a dual basis for cohomology, defined by

$$\int_{c_A} \nu^B = \delta_A^B, \quad (8.22)$$

then we have

$$\int_{\mathcal{B}} \nu^A \wedge \nu^B = I^{AB} = I^{BA}, \quad (8.23)$$

where I^{AB} is the inverse of the intersection matrix, $I^{AB} I_{BC} = \delta_C^A$. Suppose that the harmonic parts of the Maxwell fields, F^I , are given by

$$F_{harm}^I = \sigma_A^I \nu^A, \quad (8.24)$$

then (6.32) with the normalization set in (6.36) implies that

$$\sigma_A^I = 4\pi \Pi_{AA+1}^{(I)} = -4\pi \tilde{I}^{ij} \alpha_{A,i} k_j^I. \quad (8.25)$$

Moreover, (8.23) implies that the expression (8.8) for the mass may be rewritten

$$\begin{aligned} M &= -\frac{1}{32\pi G_5} C_{IJK} \alpha^I \int_{\mathcal{B}} F^J \wedge F^K = -\frac{1}{32\pi G_5} C_{IJK} \alpha^I I^{AB} \sigma_A^J \sigma_B^K \\ &= -\frac{\pi}{2G_5} C_{IJK} \alpha^I I^{AB} (\tilde{I}^{ij} \alpha_{A,i} k_j^J) (\tilde{I}^{k\ell} \alpha_{B,k} k_\ell^J) \\ &= -\frac{\pi}{2G_5} C_{IJK} \alpha^I \hat{I}_{ik} \tilde{I}^{k\ell} \tilde{I}^{ij} k_j^J k_\ell^J, \end{aligned} \quad (8.26)$$

where

$$\hat{I}_{ij} \equiv I^{AB} \alpha_{A,i} \alpha_{B,j}. \quad (8.27)$$

However, observe that

$$\hat{I}_{ij} \tilde{I}^{jk} \alpha_{A,k} = I^{BC} \alpha_{B,i} (\alpha_{C,j} I^{jk} \alpha_{A,k}) = I^{BC} \alpha_{B,i} I_{CA} = \alpha_{A,i}, \quad (8.28)$$

$$\hat{I}_{ij} \tilde{I}^{jk} q_k = \sum_{j=1}^N \hat{I}_{ij} = I^{AB} \alpha_{A,i} \sum_{j=1}^N \alpha_{B,j} = 0. \quad (8.29)$$

Thus $\widehat{I}_{ij} \tilde{I}^{jk}$ is the projector onto the space spanned by the $\alpha_{A,i}$ and with null vector q_k . This implies that

$$\widehat{I}_{ij} \tilde{I}^{jk} k_k^I = \tilde{k}_i^I, \quad (8.30)$$

where \tilde{k}_i^I is defined in (6.49). It then follows from (8.26) that

$$\begin{aligned} M &= -\frac{1}{32\pi G_5} C_{IJK} \alpha^I \int_{\mathcal{B}} F^J \wedge F^K \\ &= -\frac{\pi}{2G_5} C_{IJK} \alpha^I \sum_{i=1}^N \frac{\tilde{k}_i^J \tilde{k}_i^K}{q_i} \\ &= \frac{\pi}{4G_5} \alpha^I Q_I = \frac{\pi}{4G_5} (Q_1 + Q_2 + Q_3), \end{aligned} \quad (8.31)$$

where we have used (6.51) and taken $\alpha^I = +1$ so that λ^I in (8.1) does indeed fall off at infinity, which means that there are no boundary terms in the Smarr formula. Once again we reproduce the correct expression, (6.44), for the mass and this further confirms our analysis of the topology and the intersection form.

9 Conclusion

In this paper we have analyzed the regularity of a class of asymptotically flat BPS fuzzball solutions in five-dimensional supergravity and placed them in the context of the search for gravitational solitons. In particular, we find that arguments valid in four dimensions based on the Smarr formula, which exclude the existence of solitonic solutions without horizons no longer hold in five dimensions because the Smarr formula acquires extra bulk terms if the spatial manifold has non-trivial second homology.

Although we constructed solutions using ambi-polar hyper-Kähler manifolds, which reverse metric signature on hypersurfaces, the resulting space-times have everywhere non-singular Lorentz metrics. These singular hypersurfaces become, in the complete Lorentzian metric, a novel form of evanescent ergosurface for which there is no ergo-region. The space-times are globally hyperbolic and stably causal, having the topology $\mathbb{R} \times \mathcal{B}$ where \mathcal{B} is a complete, four-dimensional Riemannian manifold with non-trivial second homology. Specifically, we find that if \mathcal{B} is asymptotically Euclidean, *i.e.* $q_0 = 1$, then it has the topology of a p -fold connected sum of $(S^2 \times S^2) \# \dots \# (S^2 \times S^2)$ with a point removed and hence has Euler number $2p + 1$. If $q_0 = 0$ then \mathcal{B} has the topology of $(R^2 \times S^2) \# (S^2 \times S^2) \# \dots \# (S^2 \times S^2)$ with Euler number $2p$, where p is the number of summands. Thus every time one adds a point and an anti-point one blows up an $S^2 \times S^2$. It is, perhaps, significant that four-dimensional gravitational instantons representing p black holes, if such instantons exist, are expected to have the foregoing topology.

It may seem rather baroque to construct smooth five-dimensional solutions, by starting from families of four-dimensional singular geometries and then repairing them. One should however, remember that this approach reduces the BPS equations to a linear system and this means that not only can one construct large families of solitons but one can also analyze their moduli spaces with relative ease.

As our analysis of the Smarr formula shows, the existence of these microstate geometries depends critically upon the non-trivial topology. The existence of these solitonic geometries also seems to require the Chern-Simons interactions: The solutions are BPS and so necessarily have electric charge and smoothness requires that this must be derived from cohomological fluxes in the spatial topology. Such fluxes are intrinsically magnetic and so can only generate the electric charge through the Chern-Simons interaction.

There has also been extensive work on extremal non-BPS solutions (see, for example, [117, 118, 119, 120, 121, 105, 90, 106, 107, 122, 123, 124, 108, 109]) and our analysis of the Smarr formula is valid for any stationary solution. It would be interesting to investigate the more detailed analysis to these more general solutions and examine the role of the Chern-Simons interaction for such non-BPS solutions.

Finally, there is the question of whether the fuzzball geometries play a role in nature. Before embarking on such a discussion, one should note that there is considerable danger in trying to extract properties of general black holes from extremal solutions. On the other hand, given a BPS soliton, there are several simple ways to perturb it to obtain a near-BPS solitons. For example, one can include small, non-supersymmetric fluctuations of the bubbles or one can set the elements of the BPS geometry in motion along flat, compact directions of the BPS moduli space. It is also possible to add stable, perturbative probes that move the mass above the BPS bound [125]. These deformations do not destabilize the original background topology and so the topological support of the soliton is not simply one of extremely fine tuning and it does not evaporate the moment the object is perturbed away from extremality. It is obviously a huge jump from near-extremality to Schwarzschild black holes, but it is still intriguing to speculate about the role that solitons might play in the universe.

One of the primary motivations of the fuzzball program is to see if there are geometries that might describe a rich class of semi-classical representatives of black-hole microstates. Since the majority of the microstates of black holes could potentially involve Planck-scale details that are well beyond the validity of the supergravity approximation, it is most likely that supergravity solitons will only sample the states of the black-hole at some coarse-grained level. On the other hand, it may just be that such coarse graining will be sufficient to provide a useful, semi-classical description of black-hole thermodynamics. In this context, it is worth returning, once again, to historical precedents and remembering that the Maxwell-Boltzmann distribution and the

statistical basis of entropy came 30 years before Planck’s constant and over 50 years before the formulation of quantum mechanics.

More generally, one can interpret that earlier results about “no solitons without horizons” as saying that time-independent bosonic fields alone do not seem to be stiff enough to hold up a stationary end state of a star while the more refined result described here shows that there is a new possibility for supporting a stellar remnant: topology and magnetic fluxes. It is therefore tempting to conjecture new stellar end states, or *topological stars*, that are held up by such forces. The existence of the relevant topological terms in the action requires that space-time have at least five dimensions and that the extra dimensions must therefore begin to play a major role at the horizon scale of an apparently four-dimensional black hole.

These possibilities may seem far-fetched, but given the unlikely path that led to the discovery of macroscopic fuzzballs in supergravity, one of us at least is very tempted to believe in a seventh impossible thing before breakfast: that fuzz balls will play an important role in nature and in the description of the semi-classical microstate structure of black holes.

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References

- [1] S. D. Mathur, “The Fuzzball proposal for black holes: An Elementary review,” Fortsch. Phys. **53**, 793 (2005) [hep-th/0502050].
- [2] S. D. Mathur, “The Quantum structure of black holes,” Class. Quant. Grav. **23**, R115 (2006) [hep-th/0510180].
- [3] S. D. Mathur, “Fuzzballs and the information paradox: A Summary and conjectures,” arXiv:0810.4525 [hep-th].
- [4] O. Lunin and S. D. Mathur, “Metric of the multiply wound rotating string,” Nucl. Phys. B **610**, 49 (2001) [arXiv:hep-th/0105136].

- [5] O. Lunin, J. M. Maldacena and L. Maoz, “Gravity solutions for the D1-D5 system with angular momentum,” arXiv:hep-th/0212210.
- [6] O. Lunin and S. D. Mathur, “AdS/CFT duality and the black hole information paradox,” Nucl. Phys. B **623**, 342 (2002) [arXiv:hep-th/0109154].
- [7] A. Sen, “Extremal black holes and elementary string states,” Mod. Phys. Lett. A **10**, 2081 (1995) [arXiv:hep-th/9504147].
- [8] A. Strominger and C. Vafa, “Microscopic Origin of the Bekenstein-Hawking Entropy,” Phys. Lett. B **379**, 99 (1996) [arXiv:hep-th/9601029].
- [9] S. Giusto and S. D. Mathur, “Geometry of D1-D5-P bound states,” Nucl. Phys. B **729**, 203 (2005) [hep-th/0409067].
- [10] I. Bena and N. P. Warner, “Bubbling supertubes and foaming black holes,” Phys. Rev. D **74** (2006) 066001 [hep-th/0505166].
- [11] P. Berglund, E. G. Gimon and T. S. Levi, “Supergravity microstates for BPS black holes and black rings,” JHEP **0606** (2006) 007 [hep-th/0505167].
- [12] A. Saxena, G. Potvin, S. Giusto and A. W. Peet, “Smooth geometries with four charges in four dimensions,” JHEP **0604**, 010 (2006) [hep-th/0509214].
- [13] I. Bena and N. P. Warner, “Black holes, black rings and their microstates,” Lect. Notes Phys. **755** (2008) 1 [hep-th/0701216].
- [14] K. Skenderis and M. Taylor, “The fuzzball proposal for black holes,” Phys. Rept. **467**, 117 (2008) [arXiv:0804.0552 [hep-th]].
- [15] V. Balasubramanian, J. de Boer, S. El-Showk and I. Messamah, “Black Holes as Effective Geometries,” Class. Quant. Grav. **25**, 214004 (2008) [arXiv:0811.0263 [hep-th]].
- [16] B. D. Chowdhury and A. Virmani, “Modave Lectures on Fuzzballs and Emission from the D1-D5 System,” arXiv:1001.1444 [hep-th].
- [17] R. Serini, Euclideanità dello spazio completamente vuoto nella relatività general di Einstein *Atti della Accademia dei Lincei Ser. 5 Rendiconti* **27** (1918) 235-238
- [18] A. Einstein, “Demonstration of the non-existence of gravitational fields with a non-vanishing total mass free of singularities,” *Univ. Nac. Tucumán. Revista A* **2** 5-15 (1941)

- [19] A. Einstein and W. Pauli, “On the non-existence of regular stationary solutions of relativistic field equations,” *Ann Math* **44** 131-137 (1943)
- [20] P. Breitenlohner, D. Maison and G. W. Gibbons, “Four-Dimensional Black Holes from Kaluza-Klein Theories,” *Commun. Math. Phys.* **120** (1988) 295.
- [21] B. Carter, “Mathematical Foundations of the Theory of Relativistic Stellar and Black Hole Configurations,” in *Gravitation in Astrophysics: Cargese 1986* B. Carter and J. Hartle eds. Nato ASI series B: Physics Vol 156 , Plenum (1986)
- [22] R. Schon and S. -T. Yau, “On the Proof of the positive mass conjecture in general relativity,” *Commun. Math. Phys.* **65**, 45 (1979).
- [23] E. Witten, “A Simple Proof of the Positive Energy Theorem,” *Commun. Math. Phys.* **80**, 381 (1981).
- [24] G. T. Horowitz and J. Polchinski, “A Correspondence principle for black holes and strings,” *Phys. Rev. D* **55**, 6189 (1997) [hep-th/9612146].
- [25] T. Damour and G. Veneziano, “Self-gravitating fundamental strings and black holes,” *Nucl. Phys. B* **568**, 93 (2000) [hep-th/9907030].
- [26] I. Bena and N. P. Warner, “One ring to rule them all ... and in the darkness bind them?,” *Adv. Theor. Math. Phys.* **9**, 667 (2005) [hep-th/0408106].
- [27] F. Denef, “Supergravity flows and D-brane stability,” *JHEP* **0008**, 050 (2000) [hep-th/0005049].
- [28] C. -W. Wang, “Split Flows in Bubbled Geometries,” *JHEP* **1011**, 027 (2010) [arXiv:1005.0210 [hep-th]].
- [29] I. Bena, C. -W. Wang and N. P. Warner, “The Foaming three-charge black hole,” *Phys. Rev. D* **75**, 124026 (2007) [hep-th/0604110].
- [30] I. Bena, C.-W. Wang and N. P. Warner, “Mergers and typical black hole microstates,” *JHEP* **0611**, 042 (2006) [hep-th/0608217].
- [31] F. Denef and G. W. Moore, “Split states, entropy enigmas, holes and halos,” *JHEP* **1111**, 129 (2011) [hep-th/0702146 [HEP-TH]].
- [32] I. Bena, C.-W. Wang and N. P. Warner, “Plumbing the Abyss: Black ring microstates,” *JHEP* **0807**, 019 (2008) [arXiv:0706.3786 [hep-th]].

- [33] V. Balasubramanian, E. G. Gimon and T. S. Levi, “Four Dimensional Black Hole Microstates: From D-branes to Spacetime Foam,” JHEP **0801**, 056 (2008) [hep-th/0606118].
- [34] F. Denef, “Quantum quivers and Hall / hole halos,” JHEP **0210**, 023 (2002) [hep-th/0206072].
- [35] B. Bates and F. Denef, “Exact solutions for supersymmetric stationary black hole composites,” JHEP **1111**, 127 (2011) [hep-th/0304094].
- [36] J. de Boer, S. El-Showk, I. Messamah and D. Van den Bleeken, “Quantizing N=2 multi-center Solutions,” JHEP **0905**, 002 (2009) [arXiv:0807.4556 [hep-th]].
- [37] J. de Boer, S. El-Showk, I. Messamah and D. V. d. Bleeken, “A bound on the entropy of supergravity?,” arXiv:0906.0011 [hep-th].
- [38] I. Kanitscheider, K. Skenderis and M. Taylor, “Fuzzballs with internal excitations,” JHEP **0706**, 056 (2007) [arXiv:0704.0690 [hep-th]].
- [39] K. Skenderis and M. Taylor, “Anatomy of bubbling solutions,” JHEP **0709**, 019 (2007) [arXiv:0706.0216 [hep-th]].
- [40] R. Boscovich, *Theoria philosphiae naturalis* (1758) translated as *A Theory of Natural Philosophy* by E. Child M.I. T Press (1955)
- [41] H.A. Lorentz, “The Theory of electrons and its application to the phenomena of light and radiant heat,” 2nd edition (1915).
- [42] G. Mie, “Grundlagen einer Theorie der Materie,” *Ann Phys* **37** (1912) 511-534, **39**(1912) 1-40, **40**(1913) 1-66
- [43] M. Born and L. Infeld, “Foundations of the new field theory,” Proc. Roy. Soc. Lond. A **144** (1934) 425.
- [44] E. S. Fradkin and A. A. Tseytlin, “Nonlinear Electrodynamics from Quantized Strings,” Phys. Lett. B **163**, 123 (1985).
- [45] A. Abouelsaood, C. G. Callan, Jr., C. R. Nappi and S. A. Yost, “Open Strings in Background Gauge Fields,” Nucl. Phys. B **280**, 599 (1987).
- [46] A. Strominger, “Open p-branes,” Phys. Lett. B **383**, 44 (1996) [hep-th/9512059].
- [47] P. K. Townsend, “D-branes from M-branes,” Phys. Lett. B **373** (1996) 68 [hep-th/9512062].

- [48] C. G. Callan and J. M. Maldacena, “Brane death and dynamics from the Born-Infeld action,” Nucl. Phys. B **513** (1998) 198 [hep-th/9708147].
- [49] G. W. Gibbons, “Born-Infeld particles and Dirichlet p-branes,” Nucl. Phys. B **514** (1998) 603 [hep-th/9709027].
- [50] G. ’t Hooft, “Magnetic Monopoles in Unified Gauge Theories,” Nucl. Phys. B **79**, 276 (1974).
- [51] A. M. Polyakov, “Particle Spectrum in the Quantum Field Theory,” JETP Lett. **20**, 194 (1974) [Pisma Zh. Eksp. Teor. Fiz. **20**, 430 (1974)].
- [52] M. K. H. Kiessling, “Electromagnetic field theory without divergence problems. 1. The Born legacy,” math-ph/0306076.
- [53] M. K. H. Kiessling, “Electromagnetic field theory without divergence problems. 2. A Least invasively quantized theory,” math-ph/0311034.
- [54] M. K. -H. Kiessling, “On the motion of point defects in relativistic fields,” arXiv:1107.2286 [math-ph].
- [55] M. K. -H. Kiessling, “On the Quasi-Linear Elliptic PDE $-\nabla \cdot (\nabla u / \sqrt{1 - |\nabla u|^2}) = 4\pi \sum_k a_k \delta_{s_k}$ in Physics and Geometry,” Commun. Math. Phys. **314** (2012) 509 [arXiv:1107.4126 [math-ph]].
- [56] M. K. H. Kiessling, “Convergent perturbative power series solution of the stationary Maxwell-Born-Infeld field equations with regular sources,” J. Math. Phys. **52** (2011) 022902.
- [57] M. K. H. Kiessling “Some uniqueness results for stationary solutions to the Maxwell-Born-Infeld field equations and their physical consequences,” Phys Lett A **375** (2011) 3925-3930
- [58] J. Speck “The Nonlinear Stability of the Trivial Solution to the Maxwell-Born-Infeld System,” J. Math. Phys. **53** (2012) 083703 [arXiv:1008.5018 [math-ph]].
- [59] A. Chernitskii, “Nonlinear Electrodynamics with Singularities (Modernized Born-Infeld Electrodynamics)” *Helv.Phys.Acta* **71**(1998) 274-287, arXiv hep-th/ 9705075
- [60] D. Chruscinski, “Point charge in the Born-Infeld electrodynamics,” Phys. Lett. A **240** (1998) 8 [hep-th/9712161].
- [61] A.Lichnerowicz, *Théories Relativiste de la Gravitation et de l’Electromagnétisme* Paris, Masson (1955)

- [62] A. Einstein, L. Infeld and B. Hoffmann, “The Gravitational Equations and the Problem of Matter,” *Ann Math* **39** (1938) 65-100
- [63] S. R. Coleman, “Classical Lumps and their Quantum Descendents,” *Subnucl. Ser.* **13** (1977) 297.
- [64] P. Hajicek, “Quantum Wormholes. 1. Choice Of The Classical Solution,” *Nucl. Phys. B* **185**, 254 (1981).
- [65] G.W. Gibbons Soliton States and Central Charges in Extended Supergravity Theories *Proceedings of the Heisenberg Memorial Symposium 1981* eds. P Breitenlohner & H. P. Durr *Springer Lecture Notes in Physics* **160** 145–151 (1982)
- [66] G. W. Gibbons, *Lect. Notes Phys.* **383** (1991) 110 [arXiv:1109.3538 [gr-qc]].
- [67] M. E. Ortiz, “Curved space magnetic monopoles,” *Phys. Rev. D* **45** (1992) 2586.
- [68] S. R. Coleman, “Q Balls,” *Nucl. Phys. B* **262**, 263 (1985) [Erratum-ibid. *B* **269**, 744 (1986)].
- [69] R. Friedberg, T. D. Lee and Y. Pang, “Scalar Soliton Stars And Black Holes,” *Phys. Rev. D* **35** (1987) 3658.
- [70] R. G. McLenaghan and N. Tariq “A new solution of the Einstein-Maxwell equations” *Jour. Math. Phys.* **16** (1975) 2306-2312
- [71] B. Carter, Black Hole Equilibrium States, in *Black Holes Les Astres Occlus: Les Houches, Aout 1972, Cours de L'Ecole D't de Physique Thorique, [23me Session]*
- [72] S. W. Hawking, “Black hole explosions,” *Nature* **248** (1974) 30.
- [73] G.W. Gibbons, The Multiplet Structure of Solitons in the N=2 Supergravity Theory *Quantum structure of space and time* eds. M J Duff & C J Isham (Cambridge University Press) 317–321 (1983) *Talk given at Workshop on Quantum Gravity, London, Eng., Aug 1981*
- [74] G. W. Gibbons and C. M. Hull, ‘A Bogomolny Bound for General Relativity and Solitons in N=2 Supergravity,” *Phys. Lett. B* **109** (1982) 190.
- [75] The Bogomolny Inequality for Einstein-Maxwell Theory *Monopoles in Quantum Field Theory* eds. N Craigie, P Goddard & W Nahm (World Scientific, Singapore) 137–138 (1982)
- [76] R. M. Schon and S. -T. Yau, “Proof Of The Positive Action Conjecture In Quantum Relativity,” *Phys. Rev. Lett.* **42**, 547 (1979).

- [77] G. W. Gibbons, Solitons And Black Holes In Four-dimensions, Five-dimensions,” In *De Vega, H.j. (Ed.), Sanchez, N. (Ed.): Field Theory, Quantum Gravity and Strings*, 46-59
- [78] R. D. Sorkin, ‘Kaluza-Klein Monopole, Phys. Rev. Lett. **51** (1983) 87.
- [79] D. J. Gross and M. J. Perry, ‘Magnetic Monopoles in Kaluza-Klein Theories,” Nucl. Phys. B **226** (1983) 29.
- [80] G. W. Gibbons and M. J. Perry, ‘Soliton - Supermultiplets And Kaluza-klein Theory Nucl. Phys. B **248** (1984) 629.
- [81] P. Bizon, T. Chmaj and B. G. Schmidt, Critical behavior in vacuum gravitational collapse in 4+1 dimensions,” Phys. Rev. Lett. **95** (2005) 071102 [gr-qc/0506074].
- [82] P. Bizon, T. Chmaj, A. Rostworowski, B. G. Schmidt and Z. Tabor, On vacuum gravitational collapse in nine dimensions,” Phys. Rev. D **72** (2005) 121502 [gr-qc/0511064].
- [83] P. Bizon, T. Chmaj and G. Gibbons, Nonlinear perturbations of the Kaluza-Klein monopole,” Phys. Rev. Lett. **96** (2006) 231103 [gr-qc/0604043].
- [84] P. Bizon, T. Chmaj, G. W. Gibbons and C. N. Pope, Gravitational Solitons and the Squashed Seven-Sphere,” Class. Quant. Grav. **24** (2007) 4751 [hep-th/0701190].
- [85] J. P. Gauntlett, R. C. Myers and P. K. Townsend, “Black holes of $D = 5$ supergravity,” Class. Quant. Grav. **16** (1999) 1 [hep-th/9810204].
- [86] T. Shiromizu, S. Ohashi and R. Suzuki, “A no-go on strictly stationary spacetimes in four/higher dimensions,” Phys. Rev. D **86**, 064041 (2012) [arXiv:1207.7250 [gr-qc]].
- [87] C. W. Misner and J. A. Wheeler, ‘Classical physics as geometry: Gravitation, electromagnetism, unquantized charge, and mass as properties of curved empty space, Annals Phys. **2** (1957) 525.
- [88] J. P. Gauntlett, J. B. Gutowski, C. M. Hull, S. Pakis and H. S. Reall, “All supersymmetric solutions of minimal supergravity in five- dimensions,” Class. Quant. Grav. **20**, 4587 (2003) [hep-th/0209114].
- [89] J. P. Gauntlett and J. B. Gutowski, “General concentric black rings,” Phys. Rev. D **71**, 045002 (2005) [hep-th/0408122].
- [90] I. Bena, S. Giusto, C. Ruef and N. P. Warner, “Supergravity Solutions from Floating Branes,” JHEP **1003** (2010) 047 [arXiv:0910.1860 [hep-th]].

- [91] S. Hollands, “Black hole uniqueness theorems and new thermodynamic identities in eleven dimensional supergravity,” *Class. Quant. Grav.* **29**, 205009 (2012) [arXiv:1204.3421 [gr-qc]].
- [92] S. Hollands and A. Ishibashi, “Black hole uniqueness theorems in higher dimensional space-times,” *Class. Quant. Grav.* **29**, 163001 (2012) [arXiv:1206.1164 [gr-qc]].
- [93] S. Hawking, “The Existence of cosmic time functions,” *Proc. Roy. Soc. Lond. A* **308** (1968) 433.
- [94] I. Bena, P. Kraus and N. P. Warner, “Black rings in Taub-NUT,” *Phys. Rev. D* **72**, 084019 (2005) [hep-th/0504142].
- [95] G. W. Gibbons, D. Kastor, L. A. J. London, P. K. Townsend and J. H. Traschen, “Supersymmetric selfgravitating solitons,” *Nucl. Phys. B* **416**, 850 (1994) [hep-th/9310118].
- [96] W. A. Sabra, “General BPS black holes in five-dimensions,” *Mod. Phys. Lett. A* **13**, 239 (1998) [hep-th/9708103].
- [97] R. C. Myers, M. J. Perry and , “Black Holes in Higher Dimensional Space-Times,” *Annals Phys.* **172**, 304 (1986).
- [98] A. W. Peet, “TASI lectures on black holes in string theory,” hep-th/0008241.
- [99] H. Elvang, R. Emparan, D. Mateos and H. S. Reall, “Supersymmetric black rings and three-charge supertubes,” *Phys. Rev. D* **71**, 024033 (2005) [hep-th/0408120].
- [100] F. Denef, D. Gaiotto, A. Strominger, D. Van den Bleeken and X. Yin, “Black Hole Deconstruction,” *JHEP* **1203**, 071 (2012) [hep-th/0703252 [HEP-TH]].
- [101] J. de Boer, F. Denef, S. El-Showk, I. Messamah and D. Van den Bleeken, “Black hole bound states in $AdS(3) \times S^{*2}$,” *JHEP* **0811**, 050 (2008) [arXiv:0802.2257 [hep-th]].
- [102] I. Bena, N. Bobev, S. Giusto, C. Ruef and N. P. Warner, “An Infinite-Dimensional Family of Black-Hole Microstate Geometries,” *JHEP* **1103**, 022 (2011) [Erratum-ibid. **1104**, 059 (2011)] [arXiv:1006.3497 [hep-th]].
- [103] M. K. Prasad, “Equivalence of Eguchi-Hanson metric to two-center Gibbons-Hawking metric,” *Phys. Lett. B* **83**, 310 (1979).
- [104] V. Jejjala, O. Madden, S. F. Ross and G. Titchener, “Non-supersymmetric smooth geometries and D1-D5-P bound states,” *Phys. Rev. D* **71**, 124030 (2005) [arXiv:hep-th/0504181].

- [105] I. Bena, S. Giusto, C. Ruef and N. P. Warner, “A (Running) Bolt for New Reasons,” JHEP **0911**, 089 (2009) [arXiv:0909.2559 [hep-th]].
- [106] N. Bobev and C. Ruef, “The Nuts and Bolts of Einstein-Maxwell Solutions,” JHEP **1001**, 124 (2010) [arXiv:0912.0010 [hep-th]].
- [107] G. Dall’Agata, S. Giusto and C. Ruef, “U-duality and non-BPS solutions,” JHEP **1102**, 074 (2011) [arXiv:1012.4803 [hep-th]].
- [108] G. Bossard and C. Ruef, “Interacting non-BPS black holes,” Gen. Rel. Grav. **44**, 21 (2012) [arXiv:1106.5806 [hep-th]].
- [109] G. Bossard, “Octonionic black holes,” JHEP **1205**, 113 (2012) [arXiv:1203.0530 [hep-th]].
- [110] G. Compere, K. Copsey, S. de Buyl and R. B. Mann, ‘Solitons in Five Dimensional Minimal Supergravity: Local Charge, Exotic Ergoregions, and Violations of the BPS Bound, JHEP **0912** (2009) 047 [arXiv:0909.3289 [hep-th]].
- [111] Z. W. Chong, M. Cvetič, H. Lu and C. N. Pope, “Non-extremal rotating black holes in five-dimensional gauged supergravity,” Phys. Lett. B **644**, 192 (2007) [hep-th/0606213].
- [112] G. W. Gibbons and C. N. Pope, \mathbb{CP}^2 As A Gravitational Instanton, Commun. Math. Phys. **61** (1978) 239.
- [113] G. W. Gibbons and S. W. Hawking, “Classification of Gravitational Instanton Symmetries,” Commun. Math. Phys. **66**, 291 (1979).
- [114] A. L. Yuille, Israel-wilson Metrics In The Euclidean Regime, *Class. Quant. Grav.* **4** (1987) 1409.
- [115] B. Whitt, Israel-wilson Metrics,’ *Annals Phys.* **161** (1985) 241.
- [116] M. Dunajski and S. A. Hartnoll, Einstein-Maxwell gravitational instantons and five dimensional solitonic strings, *Class. Quant. Grav* **24** (2007) 1841 [hep-th/0610261]
- [117] E. G. Gimon, F. Larsen and J. Simon, “Black holes in Supergravity: The Non-BPS branch,” JHEP **0801**, 040 (2008) [arXiv:0710.4967 [hep-th]].
- [118] K. Goldstein and S. Katmadas, “Almost BPS black holes,” JHEP **0905**, 058 (2009) [arXiv:0812.4183 [hep-th]].
- [119] I. Bena, G. Dall’Agata, S. Giusto, C. Ruef and N. P. Warner, “Non-BPS Black Rings and Black Holes in Taub-NUT,” JHEP **0906**, 015 (2009) [arXiv:0902.4526 [hep-th]].

- [120] S. Bellucci, S. Ferrara, M. Gunaydin and A. Marrani, “SAM Lectures on Extremal Black Holes in d=4 Extended Supergravity,” Springer Proc. Phys. **134**, 1 (2010) [arXiv:0905.3739 [hep-th]].
- [121] I. Bena, S. Giusto, C. Ruef and N. P. Warner, “Multi-Center non-BPS Black Holes - the Solution,” JHEP **0911**, 032 (2009) [arXiv:0908.2121 [hep-th]].
- [122] N. Bobev, B. Niehoff and N. P. Warner, “Hair in the Back of a Throat: Non-Supersymmetric Multi-Center Solutions from Kähler Manifolds,” JHEP **1110**, 149 (2011) [arXiv:1103.0520 [hep-th]].
- [123] G. Dall’Agata, “Black holes in supergravity: flow equations and duality,” Springer Proceedings in Physics 142 (2013) 1-46 [arXiv:1106.2611 [hep-th]].
- [124] O. Vasilakis and N. P. Warner, “Mind the Gap: Supersymmetry Breaking in Scaling, Microstate Geometries,” JHEP **1110**, 006 (2011) [arXiv:1104.2641 [hep-th]].
- [125] I. Bena, A. Puhm and B. Vercnocke, “Non-extremal Black Hole Microstates: Fuzzballs of Fire or Fuzzballs of Fuzz ?,” JHEP **1212**, 014 (2012) [arXiv:1208.3468 [hep-th]].